Applications of Mixed Hodge Module Theory

混合霍奇模的应用

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1. Introduction
   - Mixed Hodge Module Theory
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Saito's Theory

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- Machinery: six functors + weight formalism + Hodge package
- Moral: mixed Hodge module theory over $\mathbb{C}$ parallels mixed l-adic sheaf theory over a field with characteristic $p > 0$.

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  4. It "lifts" decomposition: $HM(X, \omega)$ is semi-simple (structure theorem); direct image theorem for proper morphism, which implies BBDG decomposition theorem for perverse sheaves.

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  - (v) Zucker: global results (L^2-cohomology).

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  - (Real) Lie groups: [ABV12], [AVLTVJ12], [SV12], [DV22].
  - Koszul duality: [AK14].
  - Categorical action of $\mathfrak{sl}_2$: [CDK16].

Definition

Let $\lambda \in \mathfrak{t}^*$ be arbitrary. Then the Verma module $V(\lambda)$ admits a decreasing filtration $V(\lambda)^\bullet$ by submodules, such that

- (i) $V(\lambda)^i = 0$ for all sufficiently large $i >> 0$.
- (ii) $V(\lambda)^0 = V(\lambda)$ and $V(\lambda)^1 = N(\lambda) :=$ the unique maximal submodule of $V(\lambda)$.
- (iii) Each graded piece $V(\lambda)^i / V(\lambda)^{i+1}$ has a non-degenerate contra-variant form.
- (iv) The formal characters satisfy

$$\sum_{i>0} \mathrm{ch}(V(\lambda)^i) = \sum_{\substack{\alpha>0 \\ s_\alpha \cdot \lambda < \lambda}} \mathrm{ch}(V(s_\alpha \cdot \lambda))$$

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- Given a deformation direction $\gamma \in \mathfrak{t}^*$, one can consider the deformed Verma module $V_{\mathbb{C}[s]}(\lambda)$ which is a $(\mathfrak{g}, \mathbb{C}[s])$-bimodule generated by a highest weight vector v_λ satisfying
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$$h \cdot v_\lambda = (\lambda(h) + s\gamma(h))v_\lambda, \forall h \in \mathfrak{t}.$$
- Contravariant form: $\mathbb{C}[z]$-bilinear contra-variant form on the deformed Verma module $V_{\mathbb{C}[z]}(\lambda)$ is non-degenerate, which specializes at $z = 0$ to the contra-variant form on $V(\lambda)$.

Construction

- Filtration: On $V_{\mathbb{C}[z]}(\lambda)$, one has a filtration by order of vanishing of the form, i.e.

$$V_{\mathbb{C}[z]}(\lambda)^i := \sum_{v\in\Gamma} V_{\lambda_s - v}(i),$$

where Γ is the set of $\mathbb{Z}^+$–linear combinations of simple roots and

$$V_{\lambda_s-v}(i) := \{v \in V_{\lambda_s-v} : (v, V_{\lambda_s-v}) \subset s^i\mathbb{C}[s]\}.$$

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- The theorem above is actually the deformation in the direction $\lambda = \rho$.

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  2. The Jantzen filtration coincides with the socle filtration.
- More precisely, let $\chi_1, \chi_2 \in \mathfrak{t}^*_{\mathbb{Q}}$ be regular weights such that $V(\chi_1) \subseteq V(\chi_2)$, which means that there exists some dominant weight χ such that $\chi_i = w_i \chi$ with $w_i \in W^\chi := \{w \in W : w \cdot \chi - \chi \in \mathfrak{h}^*_{\mathbb{Z}}\}$ and $w_1 \leq w_2$. Then we have the following relation: $V(\chi_1)^i = V(\chi_1) \cap V(\chi_2)^{i+l(w_2)-l(w_1)}$.

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  - Gabber and Joseph [GJ81] showed that (1) implies the Kazhdan-Lusztig conjectures on multiplicities of simple modules in Verma modules.
- Building on the work of Gabber and Joseph, Barbasch [Bar83] showed that (1) implies (2) in a purely algebraic approach.

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- Localization theorem: Verma modules + Jantzen filtration $\to$ filtered $\mathscr{D}$−modules.

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- Part (2) is proved by a pointwise purity argument.

Dictionary

Repsentations of $\mathfrak{g}$	$\mathscr{D}_{\mathscr{B}}$-modules	l-adic sheaves
$\mathrm{Mod}(\mathfrak{g})_\chi$	$\mathrm{Mod}(\mathscr{D}^\chi_{\mathscr{B}})$	$\mathrm{Perv}(\tilde{\mathscr{B}}, \overline{\mathbb{Q}_l})_\chi$
$\mathrm{Mod}_K(\mathfrak{g})_\chi$	$Mod_K(\mathscr{D}^\chi_{\mathscr{B}})$	$\mathrm{Perv}_K(\tilde{\mathscr{B}}, \overline{\mathbb{Q}_l})_\chi$
Verma module $V(w_0 w \cdot \lambda)$	$j_!(\mathscr{L}_{\lambda, X_w})$	$j_!(\mathscr{L}_{\lambda, \tilde{X}_w})[\dim \tilde{X}_w]$
irreducible module $L(w_0 w \cdot \lambda)$	$j_{!*}(\mathscr{L}_{\lambda, X_w})$	$j_{!*}(\mathscr{L}_{\lambda, \tilde{X}_w})[\dim \tilde{X}_w]$
deformed weight $\lambda + s\rho$	function $\varphi : \overline{X_w} \to \mathbb{A}^1$ and nearby cycles	the same
multiplication by s	log monodromy s	the same
Jantzen filtration on Verma module	Jantzen filtration on $j_!$ (up to a shift)	weight filtration on $j_!$
contravariant form	$j_!(V \times I_\varphi^{(n)}) \to j_*(V \times I_\varphi^{(n)})$	the same

图 1: Dictionary for Jantzen conjecture

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- To use [BBDG18], one needs the condition "of geometric origin". One can only prove Verma modules with rational characters are of geometric origin, [BB93, Lemma 2.6.5]

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- However, we don't need this condition if we use Mixed Hodge module theory.
- We can remember the polarization via mixed Hodge Module theory.
- We use mixed Hodge modules with $\mathbb{C}$-coefficients.

Main Theorem

Fix $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, a K-orbit $Q \subseteq B$ and an irreducible K-equivariant λ-twisted flat bundle $\mathcal{V}$ on Q. Let S be a polarization of $\mathcal{V}$. Then for all n, $Gr^J_{-n} j_! \mathcal{V}$ is a pure Hodge module of weight $d - n$, and the form

$$s^{-n} Gr^J_{-n}(S) : Gr^J_{-n} j_! \mathcal{V} \xrightarrow{\cong} (Gr^J_{-n} j_! \mathcal{V})^h(-d+n)$$

is a polarization, where $(-)^h$ denotes the Hermitian dual and $d = \dim H + \dim Q$.

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- $\overline{\mathscr{M}}$ to denote the complex conjugation of $\mathscr{D}$-module $\mathscr{M}$,
- Hermitian dual M^h is defined to be the unique regular holonomic $\mathscr{D}_X$–module such that $(M^h)^{an} = \mathscr{H}\mathrm{om}_{\mathscr{D}_{\overline{X}}}(\overline{M}, \mathtt{D}b_{\overline{X}})$ (see [Kas87]).

Review: complex mixed Hodge modules

Now a polarized complex mixed Hodge module $\mathscr{M}$ consists of the following data:

- a triple $(\mathscr{M}, F_\bullet\mathscr{M}, W_\bullet\mathscr{M})$, where $\mathscr{M}$ is a regular holonomic $\mathscr{D}_X$ module, Hodge filtration $F_\bullet$ is a good filtration by $\mathscr{O}_X$–modules, and weight filtration $W_\bullet$ is a filtration by regular holonomic $\mathscr{D}_X$-submodules;

which satisfies some sophisticated conditions omitted here. Morphisms of triples are defined covariantly in $\mathscr{M}$ and contravariantly on $\mathscr{M}'$.

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- a triple $(\mathscr{M}', F_\bullet\mathscr{M}', W_\bullet\mathscr{M}')$, similar as above;

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- a triple $(\mathscr{M}', F_\bullet\mathscr{M}', W_\bullet\mathscr{M}')$, similar as above;
- a perfect sesquilinear paring $\mathfrak{s} : \mathscr{M} \otimes \overline{\mathscr{M}'} \to \mathtt{Db}_X$ compatible with $W_\bullet$ (i.e. induces an isomorphism $\mathscr{M} \cong (\mathscr{M}')^h$ of underlying $\mathscr{D}_X$-modules);

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Twisted Mixed Hodge Modules

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- Recall that $\tilde{\mathscr{B}} \to \mathscr{B}$ is a H–torsor and $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$. We define the category of λ–twisted mixed Hodge Modules on $\mathscr{B}$, denoted by $MHM_\lambda(\mathscr{B})$ to be the full subcategory of $MHM(\tilde{\mathscr{B}})$, consisting of all the objects whose underlying $\mathscr{D}$-module is the pull-back of a λ–twisted $\mathscr{D}_{\mathscr{B}}$-module on $\mathscr{B}$.

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- If $\lambda \notin \mathfrak{h}^*_{\mathbb{R}}$, then the category $MHM_\lambda(\mathscr{B})$ (defined in a similar way) must be zero (claimed in [DV22]).

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- If $\lambda \notin \mathfrak{h}^*_{\mathbb{R}}$, then the category $MHM_\lambda(\mathscr{B})$ (defined in a similar way) must be zero (claimed in [DV22]).
- Recall that a polarized variation of Hodge structure with weight ω on X gives a Hodge module of weight $\omega + \dim X$. Note that our definition of twisted mixed Hodge modules actually view $\mathcal{V}$ as a mixed Hodge module on the H–torsor $\tilde{Q}$ of Q. Therefore, the weight of $\mathcal{V}$ is actually $\dim Q + \dim H$.

Jantzen Filtration on $\mathscr{D}$-modules

- Fix $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ and a K-orbit $Q \subseteq B$.By [BB93, Lemma 3.5.2], there exists $\varphi \in \mathbb{X}^*(H)$ and a K-invariant section $f_\varphi \in H^0(Q, L^\varphi)$ such that $f_\varphi^{-1}(0) \cap Q^- = \partial Q$.

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- As $\mathcal{O}_Q$ modules, $\mathcal{V}_{s\varphi}$ is the same as $\mathcal{V}$ (we use $f_\varphi^s m \in \mathcal{V}_{s\varphi}$ to denote the corresponding $m \in \mathcal{V}$)

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- Fix $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ and a K-orbit $Q \subseteq B$.By [BB93, Lemma 3.5.2], there exists $\varphi \in \mathbb{X}^*(H)$ and a K-invariant section $f_\varphi \in H^0(Q, L^\varphi)$ such that $f_\varphi^{-1}(0) \cap Q^- = \partial Q$.
- Now we form the $(\lambda + s\varphi)$–twisted flat bundle $\mathcal{V}_{s\varphi}$ on Q.
- As $\mathcal{O}_Q$ modules, $\mathcal{V}_{s\varphi}$ is the same as $\mathcal{V}$ (we use $f_\varphi^s m \in \mathcal{V}_{s\varphi}$ to denote the corresponding $m \in \mathcal{V}$)
- but the $\mathscr{D}_Q$–module structure is given by

$$\partial\, f_\varphi^s m = f_\varphi^s(\partial m + s\frac{\partial f_\varphi}{f_\varphi} m). \tag{1}$$

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- Our assumption on φ implies that this map is an isomorphism after inverting s.
- It induces Jantzen filtrations $J_\bullet$ on the domain and codomain defined by $J_n j_! \mathcal{V} = (j_! \mathcal{V}_{s\varphi}[[s]] \cap s^{-n} j_* \mathcal{V}_{s\varphi}[[s]])/(s)$, and $J_n j_! \mathcal{V} = (s^{-n} j_! \mathcal{V}_{s\varphi}[[s]] \cap j_* \mathcal{V}_{s\varphi}[[s]])/(s)$, and isomorphisms $s^n : Gr_n^J j_* \mathcal{V} \xrightarrow{\cong} Gr_{-n}^J j_! \mathcal{V}$.

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- Step 3: verify $s^{-n}Gr^{J}_{-n}S$ coincides with the polarization on nearby cycles.

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- Step 1 is standard.
- Step 2 is almost repeating word for word [BB93].
- The key idea is to lift the comparison theorem $\pi_f^1 \cong {}^p\psi_f^{un}$ in $\mathscr{D}-$modules setting ([Bei87]) to mixed Hodge module setting. In particualr, the deformation parameter s corresponds to the nilpotent operator s on π_f^1.

Saito's Vanishing Theorem

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- $$H^i(Z, \mathrm{gr}_p^F \mathrm{DR}(M) \otimes \mathscr{L}) = 0, \text{ for } i > 0 \text{ and } p \in \mathbb{Z},$$
$$H^i(Z, \mathrm{gr}_p^F \mathrm{DR}(M) \otimes \mathscr{L}^{-1}) = 0, \text{ for } i < 0 \text{ and } p \in \mathbb{Z}.$$

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  - $\mathrm{gr}_p^F \mathrm{DR}_X(\mathscr{O}_X) = \begin{cases} \Omega_X^{-p} \otimes \mathscr{O}_X[n+p], & \text{if } -n \leq p \leq 0, \\ 0, & \text{otherwise.} \end{cases}$

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- Step 6: connect the dots.

Step 2: duality argument

- Theorem: Let $\mathscr{M} \in HM(X, \omega)$ be a polarizable Hodge module on an n-dimensional complex manifold X. Then any polarization on $\mathscr{M}$ induces an isomorphism

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- Combining the results above, we obtain

$$H^{-i}(X, \mathrm{gr}_p^F \mathrm{DR}(\mathscr{M}))^* \cong H^i(X, \mathrm{gr}_{-p-\omega}^F \mathrm{DR}(\mathscr{M})).$$

Step 4: strictness of MHM to get vanihsing of morphism

Now using strictness of pushforward to a point, we have E_1-degeneration of the spectral sequence

$$E_1 = H^{p+q}(Y, \mathrm{gr}^F_{-p}\mathscr{M}_Y) \Rightarrow H^{p+q}(Y, \mathscr{M}_Y)$$

and hence we obtain that the morphism

$$H^i(Y, \mathrm{gr}^F_p\mathscr{M}_Y) \to H^{i+1}(Y, \mathrm{gr}^F_{p-1}\mathscr{M}_Y)$$

is zero map for all $i \in \mathbb{N}$ and $p \in \mathbb{Z}$.

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Thanks!