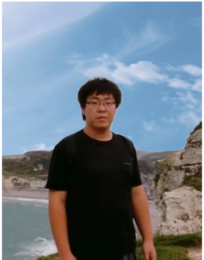
Sharp asymptotics for arm events in critical planar percolation

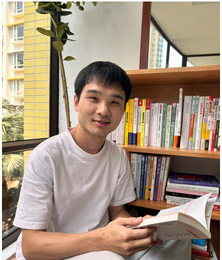
Hang Du

School of Mathematical Sciences, Peking University

June 10th, 2023

joint work with Xinyi Li (BICMR), Yifan Gao (CityUHK) and Zijie Zhuang (UPenn)





Percolation

- In physics, chemistry and material sciences: movement of fluids through porous materials.

- In mathematical physics: one of the most studied topics in statistical dynamics.
- An ideal playground for the study of **phase transition** and **criticality**.

Percolation

- In physics, chemistry and material sciences: movement of fluids through porous materials.

- In mathematical physics: one of the most studied topics in statistical dynamics.
- An ideal playground for the study of **phase transition** and **criticality**.

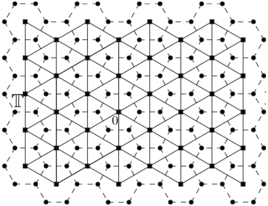
Percolation

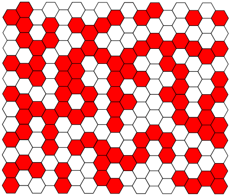
- In physics, chemistry and material sciences: movement of fluids through porous materials.

- In mathematical physics: one of the most studied topics in statistical dynamics.
- An ideal playground for the study of **phase transition** and **criticality**.

Critical planar percolation on the triangular lattice

- We consider the critical Bernoulli percolation on triangular lattice $\mathbb{T}$. For illustration, paint the hexagons in the dual lattice $\mathbb{T}^*$ instead.

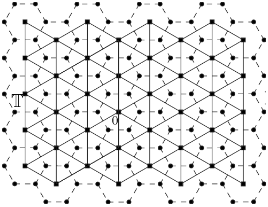


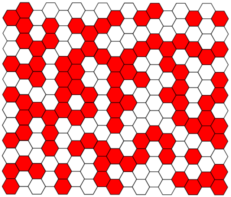


- $p_c = 1/2$ in this case.
- Great breakthrough in the last 20 years on **planar critical** percolation on $\mathbb{T}$ (thanks to the **conformal invariance**), in particular in the understanding of **scaling limit** and **critical exponents**.

Critical planar percolation on the triangular lattice

- We consider the critical Bernoulli percolation on triangular lattice $\mathbb{T}$. For illustration, paint the hexagons in the dual lattice $\mathbb{T}^*$ instead.

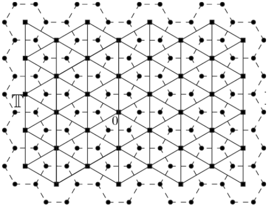


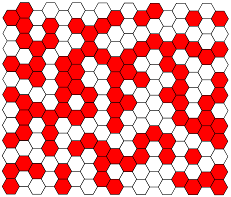


- $p_c = 1/2$ in this case.
- Great breakthrough in the last 20 years on **planar critical** percolation on $\mathbb{T}$ (thanks to the **conformal invariance**), in particular in the understanding of **scaling limit** and **critical exponents**.

Critical planar percolation on the triangular lattice

- We consider the critical Bernoulli percolation on triangular lattice $\mathbb{T}$. For illustration, paint the hexagons in the dual lattice $\mathbb{T}^*$ instead.

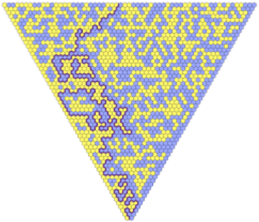


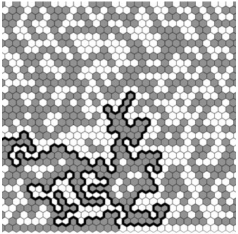


- $p_c = 1/2$ in this case.
- Great breakthrough in the last 20 years on **planar critical** percolation on $\mathbb{T}$ (thanks to the **conformal invariance**), in particular in the understanding of **scaling limit** and **critical exponents**.

A non-exhaustive list of progresses in understanding the limit of critical planar percolation

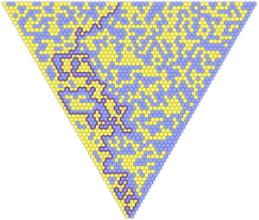
- [Schramm '00] Introduction of **Schramm-Loewner Evolution** as the conjectural scaling limit of percolation interface (and many more critical 2D models);
- [Smirnov '01] Conformal invariance of the crossing probability (aka **Cardy's formula**) and scaling limit of the interface as SLE_6;
- [Smirnov-Werner '01][Lawler-Schramm-Werner '01] Identification of the value of various **arm exponents**;

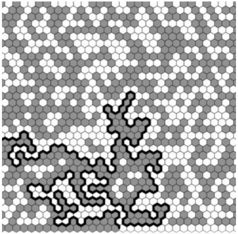




A non-exhaustive list of progresses in understanding the limit of critical planar percolation

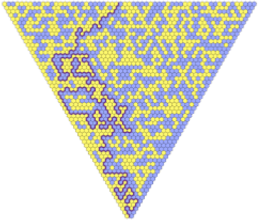
- [Schramm '00] Introduction of **Schramm-Loewner Evolution** as the conjectural scaling limit of percolation interface (and many more critical 2D models);
- [Smirnov '01] Conformal invariance of the crossing probability (aka **Cardy's formula**) and scaling limit of the interface as SLE_6;
- [Smirnov-Werner '01][Lawler-Schramm-Werner '01] Identification of the value of various **arm exponents**;

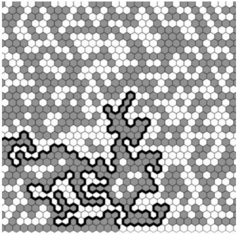




A non-exhaustive list of progresses in understanding the limit of critical planar percolation

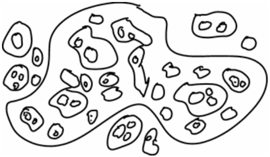
- [Schramm '00] Introduction of **Schramm-Loewner Evolution** as the conjectural scaling limit of percolation interface (and many more critical 2D models);
- [Smirnov '01] Conformal invariance of the crossing probability (aka **Cardy's formula**) and scaling limit of the interface as SLE_6;
- [Smirnov-Werner '01][Lawler-Schramm-Werner '01] Identification of the value of various **arm exponents**;

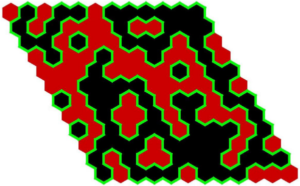




A non-exhaustive list of progresses in understanding the limit of critical planar percolation

- [Camia-Newman '06] The full scaling limit of interfaces and characterization as **Conformal Loop Ensemble**;

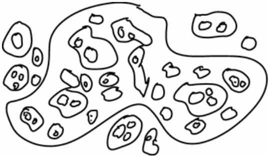


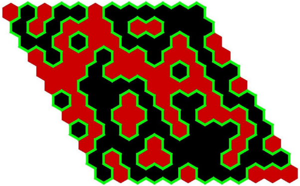


- [Garban-Pete-Schramm '10, '13, '18] Construction of scaling limits of **near-critical** and **dynamical** percolation via scaling limit of pivotal measures in the critical percolation;
- ······
- [Holden-Li-Sun '22] Scaling limit in **natural parametrization** of percolation exploration process and pivotal points.

A non-exhaustive list of progresses in understanding the limit of critical planar percolation

- [Camia-Newman '06] The full scaling limit of interfaces and characterization as **Conformal Loop Ensemble**;

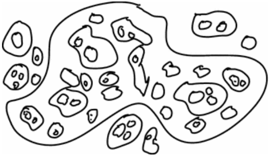


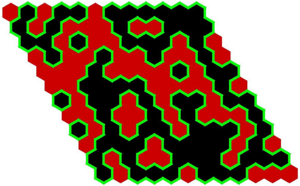


- [Garban-Pete-Schramm '10, '13, '18] Construction of scaling limits of **near-critical** and **dynamical** percolation via scaling limit of pivotal measures in the critical percolation;
- ......
- [Holden-Li-Sun '22] Scaling limit in **natural parametrization** of percolation exploration process and pivotal points.

A non-exhaustive list of progresses in understanding the limit of critical planar percolation

- [Camia-Newman '06] The full scaling limit of interfaces and characterization as **Conformal Loop Ensemble**;





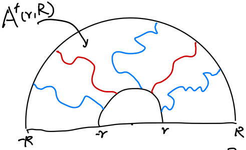
- [Garban-Pete-Schramm '10, '13, '18] Construction of scaling limits of **near-critical** and **dynamical** percolation via scaling limit of pivotal measures in the critical percolation;
- ......
- [Holden-Li-Sun '22] Scaling limit in **natural parametrization** of percolation exploration process and pivotal points.

Arm events

Definition

- *An* **arm** *is a self-avoiding path of nearest-neighbor hexagons of the same color;*
- $A^+(r, R)$*: half-annulus of inner- and outer-radius* r *and* R*;*
- *The* **half-plane** j*-arm event:*

$$\mathcal{B}_j(r, R) := \{\exists j \text{ disjoint arms crossing } A^+(r, R)\};$$



half-plane 5-arm event $\mathcal{B}_5(r,R)$

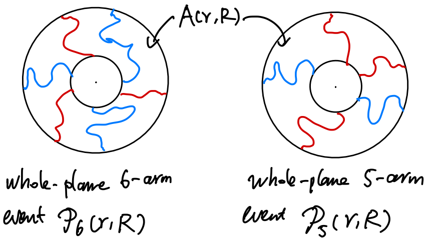
More arm events

Definition

- $A(r, R)$: *annulus of inner- and outer-radius r and R;*
- *The* **whole-plane** *(polychromatic) j-arm event:*

$$\mathcal{P}_j(r, R) := \{\exists j \text{ disjoint arms NOT all of the same color (except } j = 1) \text{ crossing } A(r, R)\};$$

- $\mathcal{A}_j(r, R) \subset \mathcal{P}_j(r, R)$: *the color sequence is alternative.*



Arm events and percolation

Arm events are central objects of interest for the study of critical (and near-critical) planar percolation.

- Whole-plane events:
  - one-arm: the cluster containing the origin;
  - two-arm: the interface, aka the exploration process;
  - four-arm event: pivotal points, correlation length, near-critical percolation, dynamical percolation;
  - $\cdots\cdots$
- Half-plane events:
  - one-arm: the cluster touching a specific point on the boundary;
  - two-arm: the hitting of the exploration process on the boundary;
  - $\cdots\cdots$

Arm exponents via the knowledge of SLE_6

Theorem (Werner-Smirnov '01)

Half-plane *exponents: for any* $j \geq 1$,

$$\mathbf{P}[\mathcal{B}_j(r,R)] = R^{-j(j+1)/6+o(1)}.$$

Whole-plane *exponents: for any* $j > 1$,

$$\mathbf{P}[\mathcal{P}_j(r,R)] = R^{-(j^2-1)/12+o(1)}.$$

Theorem (Lawler-Schramm-Werner '01)

$$\mathbf{P}[\mathcal{P}_1(r,R)] = R^{-5/48+o(1)}.$$

The quest for improvement of arm probabilities

As long as there is "$o(1)$" in the exponent of asympototics for arm probability, one is not entirely satisfied. In the proceedings of ICM 2006, Oded Schramm proposed the following

Problem (3.1)

Improve the estimates $R^{-5/48+o(1)}$ and $R^{-5/4+o(1)}$ mentioned above[a] *(as well as other similar estimates) to more precise formulas. It would be especially nice to obtain estimates that are sharp up to multiplicative constants.*

[a] i.e. arm probability asymptotics for $j = 1$ and $j = 4$ in the whole-plane case

Cited from "Conformally invariant scaling limits: an overview and a collection of problems" by Oded Schramm, *ICM proceedings*, 2006.

A small step forward

Some easy up-to-constants estimates: in the half-plane case,

$$\mathbf{P}[\mathcal{B}_2(1,R)] \asymp R^{-1} \quad \text{and} \quad \mathbf{P}[\mathcal{B}_3(1,R)] \asymp R^{-2};$$

in the whole-plane case,

$$\mathbf{P}[\mathcal{P}_5(1,R)] \asymp R^{-2}.$$

Improvements for other arm probability asymptotics are much more difficult.

- Mendelson, Nachmias and Watson obtained a rate of convergence for the Cardy's formula[1], and improved half-plane 1-arm asymptotics:

Theorem (Mendelson-Nachmias-Watson '14)

$$\mathbf{P}[\mathcal{B}_1(1,R)] = e^{O(\sqrt{\log\log R})} R^{-1/3} = (\log R)^{O(1/\sqrt{\log\log R})} R^{-1/3}.$$

[1] Also independently obtained in [Binder-Chayes-Lei '15].

A small step forward

Some easy up-to-constants estimates: in the half-plane case,

$$\mathbf{P}[\mathcal{B}_2(1,R)] \asymp R^{-1} \quad \text{and} \quad \mathbf{P}[\mathcal{B}_3(1,R)] \asymp R^{-2};$$

in the whole-plane case,

$$\mathbf{P}[\mathcal{P}_5(1,R)] \asymp R^{-2}.$$

Improvements for other arm probability asymptotics are much more difficult.

- Mendelson, Nachmias and Watson obtained a rate of convergence for the Cardy's formula[1], and improved half-plane 1-arm asymptotics:

Theorem (Mendelson-Nachmias-Watson '14)

$$\mathbf{P}[\mathcal{B}_1(1,R)] = e^{O(\sqrt{\log\log R})} R^{-1/3} = (\log R)^{O(1/\sqrt{\log\log R})} R^{-1/3}.$$

[1] Also independently obtained in [Binder-Chayes-Lei '15].

Sharp asymptotics for half-plane arm probabilities

In the half-plane case, we are now able to give sharp asymptotics for arm probabilities.

Theorem (D.-Gao-Li-Zhuang '22)

For any $j \geq 1$, for any $r \geq r_0(j)$, $\exists C = C(r),\ c = c(r)$ s.t.

$$\mathbf{P}[\mathcal{B}_j(r, R)] = CR^{-j(j+1)/6}\big(1 + O(R^{-c})\big).$$

In particular, one can take $r_0(j) = 1$ for $j = 1, 2, 3$.

- The requirement that $r \geq r_0$ devotes to ensure the arm probability $\mathbf{P}[\mathcal{B}_j(r, R)] > 0$.

Sharp asymptotics for half-plane arm probabilities

In the half-plane case, we are now able to give sharp asymptotics for arm probabilities.

Theorem (D.-Gao-Li-Zhuang '22)

For any $j \geq 1$, for any $r \geq r_0(j)$, $\exists C = C(r),\ c = c(r)$ s.t.

$$\mathbf{P}[\mathcal{B}_j(r,R)] = CR^{-j(j+1)/6}\big(1 + O(R^{-c})\big).$$

In particular, one can take $r_0(j) = 1$ for $j = 1, 2, 3$.

- The requirement that $r \geq r_0$ devotes to ensure the arm probability $\mathbf{P}[\mathcal{B}_j(r,R)] > 0$.

Up-to-constant estimates for whole-plane case

For whole-plane arm events, we can give sharp asymptotics of probabilities of alternating arm events, as well as up-to-constant estimates for probabilities of polychromatic arm events:

Theorem (D.-Gao-Li-Zhuang '22)

For any $j \geq 2$ and some $r \geq r_0(j)$, $\forall r \geq r_0$, the following hold:
For the alternating arm event $A_j(r, R)$, $\exists C' = C'(r)$ s.t.

$$\mathbf{P}[\mathcal{A}_j(r, R)] = (C' + o(1)) R^{-(j^2-1)/12};$$

For polychromatic arm event $\mathcal{P}_j(r, R)$,

$$\mathbf{P}[\mathcal{P}_j(r, R)] \asymp R^{-(j^2-1)/12}.$$

In particular, one can take $r_0(j) = 1$ for $j = 2, \cdots, 6$.

Up-to-constant estimates for whole-plane case

For whole-plane arm events, we can give sharp asymptotics of probabilities of alternating arm events, as well as up-to-constant estimates for probabilities of polychromatic arm events:

Theorem (D.-Gao-Li-Zhuang '22)

For any $j \geq 2$ *and some* $r \geq r_0(j)$, $\forall r \geq r_0$, *the following hold:*
For the alternating arm event $A_j(r, R)$, $\exists C' = C'(r)$ *s.t.*

$$\mathbf{P}[\mathcal{A}_j(r, R)] = (C' + o(1)) R^{-(j^2-1)/12};$$

For polychromatic arm event $\mathcal{P}_j(r, R)$,

$$\mathbf{P}[\mathcal{P}_j(r, R)] \asymp R^{-(j^2-1)/12}.$$

In particular, one can take $r_0(j) = 1$ *for* $j = 2, \cdots, 6$.

The need of a rate of convergence for discrete processes to SLE

(Still cited from Schramm, ibid.)

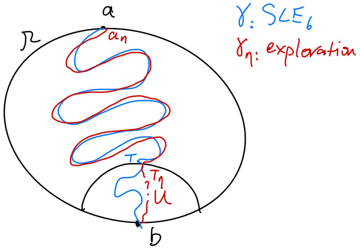
- *The difficulty in getting more precise estimates is not in the analysis of SLE. Rather, it is due to the passage between the discrete and continuous setting. Consequently, the above problem seems to be related to the following*

Problem (3.2)

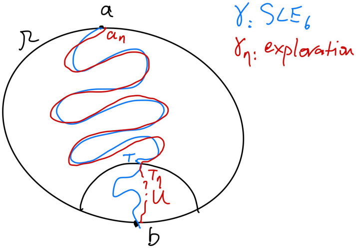
Obtain reasonable estimates for the speed of convergence of the discrete processes which are known to converge to SLE.

Power-law rate of convergence for exploration process

- Take a Jordan domain Ω and two boundary points a, b.
- For $\eta > 0$, let $(\Omega_\eta, a_\eta, b_\eta)$ be the η-discretization of Ω by $\eta\mathbb{T}^*$, along with the discrete approximation of the marked points.
- Consider critical face percolation on $\eta\mathbb{T}^*$ with Dobrushin boundary condition. Let γ_η be the exploration process from a_η to b_η and a chordal SLE_6 γ in Ω from a to b.
- Given open $U \subset \Omega$, such that $a \notin U$ and $b \in U$, let T_η (resp. T) be the first time that γ_η (resp. γ) enters U_η (resp. U).



Power-law rate of convergence for exploration process



Theorem (Binder-Richards '21)

For any $\eta > 0$, there is $u = u(\Omega, a, b, U) > 0$ and a coupling $\mathbf{P}$ *of γ_η and γ such that*

$$\mathbf{P}\left[d\left(\gamma_\eta|_{[0,T_\eta]}, \gamma|_{[0,T]}\right) > \eta^u\right] < O(\eta^u)\,,$$

where d is the up-to-reparametrization metric between curves.

Proof strategy overview

- Even equipped with the convergence rate of percolation exploration path to SLE_6, the derivation of main results is still far from trivial, since the connection between **microscopic** and **macroscopic scales** are considered;
- The **convergence rate** result allows us to further encode information between **mesoscopic** and **macroscopic scales**;
- We use **discrete coupling** techniques to encode the information between **microscopic** and **mesoscopic scales**;
- Finally, we apply a **"functional equation" trick** to put everything together and reach the sharp estimates via an **abstract approach**.

Proof strategy overview

- Even equipped with the convergence rate of percolation exploration path to SLE_6, the derivation of main results is still far from trivial, since the connection between **microscopic** and **macroscopic scales** are considered;
- The **convergence rate** result allows us to further encode information between **mesoscopic** and **macroscopic scales**;
- We use **discrete coupling** techniques to encode the information between **microscopic** and **mesoscopic scales**;
- Finally, we apply a **"functional equation" trick** to put everything together and reach the sharp estimates via an **abstract approach**.

Proof strategy overview

- Even equipped with the convergence rate of percolation exploration path to SLE_6, the derivation of main results is still far from trivial, since the connection between **microscopic** and **macroscopic scales** are considered;
- The **convergence rate** result allows us to further encode information between **mesoscopic** and **macroscopic scales**;
- We use **discrete coupling** techniques to encode the information between **microscopic** and **mesoscopic scales**;
- Finally, we apply a **"functional equation" trick** to put everything together and reach the sharp estimates via an **abstract approach**.

Proof strategy overview

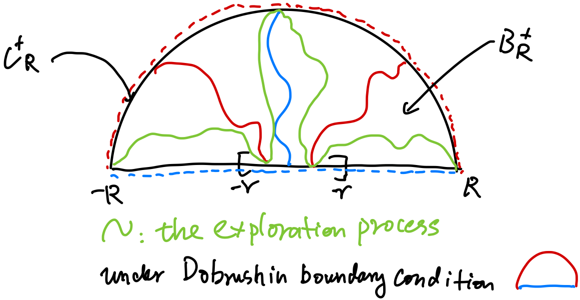
- Even equipped with the convergence rate of percolation exploration path to SLE_6, the derivation of main results is still far from trivial, since the connection between **microscopic** and **macroscopic scales** are considered;
- The **convergence rate** result allows us to further encode information between **mesoscopic** and **macroscopic scales**;
- We use **discrete coupling** techniques to encode the information between **microscopic** and **mesoscopic scales**;
- Finally, we apply a **"functional equation" trick** to put everything together and reach the sharp estimates via an **abstract approach**.

Strategy of the proof, Step 1

We focus on the "clean" half-plane case.

- Consider variants of arm events that are tailored for the application of [Binder-Richards '21]:

$$\mathcal{H}_j(r,R) := \{\exists\ j \text{ disjoint arms of alternating colors from } [-r,r]\times 0 \text{ to } C_R^+ \text{ in } B_R^+\}.$$

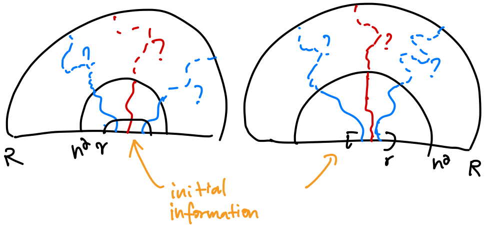


Note that this is an event that can be described by the exploration process.

Strategy of the proof, Step 2

- Use couplings of conditioned percolation configurations to relate the classical arm events to the variant defined above: for $\alpha \in (0,1)$, with universal constants,

$$\mathbf{P}[\mathcal{B}_j(r,R)|\mathcal{B}_j(r,R^\alpha)] = \mathbf{P}[\mathcal{H}_j(r,R)|\mathcal{H}_j(r,R^\alpha)]\big(1+O(R^{-c})\big),$$



and to establish a proportion between microscopic and mesoscopic arm probabilities: for $\alpha \in (0,1)$,

$$\frac{\mathbf{P}[\mathcal{H}_j(r,mR)]}{\mathbf{P}[\mathcal{H}_j(r,R)]} = \frac{\mathbf{P}[\mathcal{H}_j(R^\alpha,mR)]}{\mathbf{P}[\mathcal{H}_j(R^\alpha,R)]}\big(1+O(R^{-c})\big).$$

Strategy of the proof, Step 3

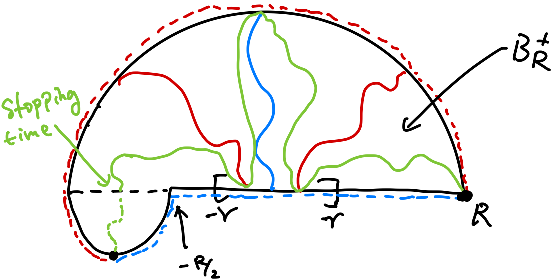
- Apply [Binder-Richards] to obtain a comparison across scales at mesoscopic level:

Proposition

There exists $c_1 > 0$ such that for all $a \in (1 - c_1, 1)$ and $m \in (1.1, 10)$

$$\mathbf{P}[\mathcal{H}_j(n^\alpha, R)] = \mathbf{P}[\mathcal{H}_j(mR^\alpha, mR)]\big(1 + O(^{-c})\big)$$

with universal constants.



Strategy of the proof, Steps 4 and 5

- Combine various asymptotic "identities" of proportions to conclude with the following one:

$$\frac{\mathbf{P}[\mathcal{B}_j(r, m^2 R)]}{\mathbf{P}[\mathcal{B}_j(r, mR)]} = \frac{\mathbf{P}[\mathcal{B}_j(r, mR)]}{\mathbf{P}[\mathcal{B}_j(r, R)]}\big(1 + O(R^{-c})\big)$$

  with uniform constants for $m \in (1.1, 10)$.
- Finally, use a "functional equation" trick to obtain the desired result from the asymptotic proportion above.

Similar strategy also appears in [L.-Shiraishi '19] to deal with sharp one-point function for 3-dimensional loop-erased random walk.

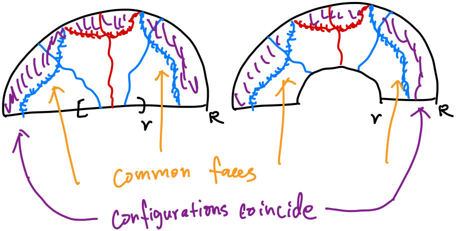
Coupling of configurations conditioned on arm events

In Step 2, we use coupling to establish estimates such as

$$\mathbf{P}[\mathcal{B}_j(r,R)|\mathcal{B}_j(r,R^\alpha)] = \mathbf{P}[\mathcal{H}_j(r,R)|\mathcal{H}_j(r,R^\alpha)]\big(1+O(R^{-c})\big),$$

Proposition

For any $j \geq 2$, there exists $\delta = \delta(j) > 0$ such that for any large r and R, there is a coupling Q of the conditional laws $\mathbf{P}[\cdot \mid \mathcal{H}_j(r,R)]$ and $\mathbf{P}[\cdot \mid \mathcal{B}_j(r,R)]$ such that if we sample (ω, ω') according to Q, then with probability at least $1-(\frac{r}{R})^\delta$, there exists a common configuration of j inner faces Θ^* around C_R^+ and ω coincides with ω' outside these faces.



Half-plane super-strong separation lemma

A crucial step in establishing the coupling is to show that interfaces between arms exhibit a "separation phenomenon" with uniformly positive probability in each scale.

Let Γ be a set of interfaces from C_u^+ to C_v^+, $u < v$. Suppose that Γ contains $j \geq 1$ interfaces and let $e^1, \cdots, e^j$ be end-edges of the interfaces in Γ on C_v^+ (in counterclockwise order). Then Γ has an **exterior quality**

$$Q_{\text{ex}}(\Gamma) := Q_{\text{ex}}^v(\Gamma) = \frac{1}{v}\,\mathrm{d}(v, e^1) \wedge \mathrm{d}(e^1, e^2) \wedge \cdots \wedge \mathrm{d}(e^j, -v).$$

Proposition (Super-strong separation lemma)

For any $j \geq 2$, there exist $M = M(j) > 1$ and $c = c(j) > 0$ such that for any $r_0 < r < u < Mu \leq R$, and any $B_u^+(\omega)$,

$$\mathbf{P}[Q_{ex}(\Gamma) > j^{-1} \mid \mathcal{B}_j(r, R), B_u^+(\omega)] > c$$

where Γ is the set of interfaces crossing $A^+(u, R)$.

A list of open questions

- Can similar strategy be applied to other models to obtain sharp asymptotics of arm probabilities, in particular the critical **FK-Ising** model (partial progress) and the harmonic explorer (seems difficult)?
- Can one improve asympotics for $\mathbf{P}[\mathcal{P}_1(1,R)]$, the whole-plane one-arm probability?
- $\cdots\cdots$

Preprint available at arXiv:2205.15901

Thank you!