de Rham epsilon factor

§1 Motivation

Beilinson's question on ε-factorization of motivic (ℓ-adic or de Rham) epsilon factors

Example k: finite field. X/k projective smooth curve.

$\mathcal{F}$: ℓ-adic sheaf on X

The global epsilon factor $\varepsilon(\mathcal{F}, X) = \det(-\mathrm{Frob}_k ; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1}$

is the constant term of the functional equation for Grothendieck L-functions of $\mathcal{F}$

$$L(X, \mathcal{F}, t) = t^{-\chi(X_{\bar{k}}, \mathcal{F})} \cdot \varepsilon(\mathcal{F}, X) \cdot L(X, D(\mathcal{F}), t^{-1})$$

$\| \det$

$\det(1 - t\,\mathrm{Frob}_k ; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1}$.

where $\chi(X_{\bar{k}}, \mathcal{F}) = \sum_i (-1)^i \dim H^i_{et}(X_{\bar{k}}, \mathcal{F})$ is the Étale Euler-Poincaré number.

$\mathrm{Frob}_k \in \mathrm{Gal}(\bar{k}/k)$ is the geometric Frobenius (inverse of arith Frob $x \mapsto x^{\# k}$)

The global epsilon factor $\varepsilon(X, \mathcal{F})$ satisfies the following product formula, conjectured by P. Deligne and proved by Laumon:

$$\varepsilon(\mathcal{F}, X) = \left(\prod_{x \in |X|} \varepsilon_x(\omega, \mathcal{F}) \right) \cdot \text{"}q^{(1-g)\operatorname{rank}\mathcal{F}}\text{"}, \quad \psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^\times \text{ fixed char.}$$

where ω is a non-vanishing meromorphic 1-form on X, $\omega \in \Omega^1_{k(X)}$.

$\varepsilon_x(\omega, \mathcal{F})$ = is the Local epsilon factor, whose existence is suggested by Langlands program, proved by Deligne or Laumon using Local ~~ep~~ Fourier transform.

The local data $\varepsilon_x(\omega, \mathcal{F})$ depends only on the restriction

$$(\omega, \mathcal{F})\Big|_{\mathrm{Spec}\, \widehat{\mathcal{O}^h_{X,x}}} \leftarrow \text{completion.}$$

Open Question

— Geometric ε-factorization of $\det R\Gamma(X, \mathcal{F})$, which gives Laumon's product formula after taking trace of Frobenius. (even open for curves)

— Higher dimensional analogues ?
We have higher class field theory, but no higher Langlands program!

$\boxed{\text{Today}}$ We present such a theory for de Rham ε-factor due to Deepam Patel. Some rank 1 theory on curves are due to Beilinson–Bloch–Esnault. See also Deligne's IHES lectures.

<u>Quick review</u>

In this talk, k always be a field of characteristic 0.
And X is a smooth variety of dimension d over k.

~~Choose d's 1-form $v_1, \dots, v_d$~~

$\underset{\text{open}}{U \subseteq} X \hookleftarrow D = X \setminus U$. $v : U \longrightarrow T^*X$ 1-form, $v \in \Omega^1_X(U)$.

Let $K(\mathcal{D}_X, v)$ be the K-theory spectrum of ~~locally finitely~~ perfect $\mathcal{D}_X$-module $\mathcal{M}$ on X such that $v(U) \cap SS(\mathcal{M}) = \phi$.

$K_D(X)$ = K-theory spectrum of coherent sheaves on X which are set theoretically supported on D.

<u>Theorem (Patel)</u> $\exists$ $K(\mathcal{D}_X, v) \xrightarrow{\mathcal{E}_{v,D}} K_D(X)$

If X is proper, we have a commutative diagram

$$\begin{tikzcd}
K(\mathcal{D}_X, v) \arrow[r, "\mathcal{E}_{v,D}"] \arrow[dr, "R\Gamma_{dR}"'] \arrow[ddrr, bend right, "\det R\Gamma_{dR}"'] & K_D(X) \arrow[d, "R\Gamma"] \arrow[ddr, bend left, "\varepsilon_{v,D}(\cancel{\det})"] & \\
 & K(k) \arrow[dr, "\det"] & \\
 & & \mathrm{Pic}^{\mathbb{Z}}(k)
\end{tikzcd}$$

$\boxed{\text{microlocal way}}$

usual Cohomological way

Thus $\det R\Gamma_{dR}(X, \mathcal{M}) \cong \varepsilon_{v,D}(\mathcal{M})$.

<u>Moreover</u>, we can choose d's 1-form $v_1, \dots, v_d$ such that D may choose to be of dimension 0.

<u>Put</u> $\underline{v} = (v_1, \dots, v_d)$.

<u>Then</u> $\det R\Gamma_{dR}(\mathcal{M}) \cong \varepsilon_{\underline{v},D}(\mathcal{M}) \cong \bigotimes_{x \in \pi_0(D)} \varepsilon_{\underline{v},x}(\mathcal{M})$.

is the promised factorization formula in higher dimensions.

§2 Construction of epsilon factors

Recall from previous ~~work~~ talks about the micro-local description of singular supports

$S' \subseteq S \subseteq T^*X$ conical closed subsets, we have a homotopy commutative diagram

$$\begin{tikzcd}
K_S(\mathcal{D}_X) \arrow[r, "gr_S"] \arrow[d] & K_S(T^*X) \arrow[d] \\
K_{S'}(\mathcal{D}_X) \arrow[r, "gr_{S'}"] \arrow[d] & K_{S'}(T^*X) \arrow[d] \\
K(\mathcal{D}_X) \arrow[r, "gr", "gr^Q"'] & K(T^*X) \\
K(X) \arrow[u, "\text{weak equivalence}"] \arrow[ur, "\pi^*"'] &
\end{tikzcd}$$

Any perfect complex $\mathcal{M}$ of $\mathcal{D}_X$-modules with $SS(\mathcal{M}) \subseteq S$, it gives rise to homotopy points

$[\mathcal{M}] \in K_S(\mathcal{D}_X)$

and $gr_S([\mathcal{M}]) \in K_S(T^*X)$.

where $K(X) \longrightarrow K(T^*X)$ is defined by pull-back by $T^*X \xrightarrow{\pi} X$

$K(X) \longrightarrow K(\mathcal{D}_X)$ is defined by $[\mathcal{F}] \longmapsto [\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}]$.

Then inverting the weak equivalences gives us a morphism

$$K(\mathcal{D}_X) \xrightarrow{gr^Q} K(T^*X).$$

- gr is defined as follows π affine and $\pi_* \mathcal{O}_{T^*X} \cong gr(\mathcal{D}_X)$.

We have an equivalence of K-theory spectra $\pi_* : K(T^*X) \longrightarrow K(gr(\mathcal{D}_X))$

$$\begin{tikzcd}
K(T^*X) \arrow[r, "\pi_*"] & K(gr(\mathcal{D}_X)) \arrow[l, bend left, "(\pi_*)^{-1}"] \\
 & K(\mathcal{D}_X) \arrow[ul] \arrow[u]
\end{tikzcd}$$

- The morphisms gr_S and $gr_{S'}$ are induced from gr

We can prove that The homotopy morphism gr^Q and gr are canonically identified.

Lemma 3.1 $Y \subseteq X$ closed subset

$S \subseteq T^*X$ conical closed subset.

$$\begin{tikzcd} v: X\setminus Y \arrow[r] \arrow[dr] & T^*X\setminus S \arrow[d] \\ & X \end{tikzcd}$$ 1-form.

then one has a commutative diagram: localized sequence

$$\begin{tikzcd}[column sep=large] K_S(\mathcal{D}_X) \arrow[r, "gr_S"] \arrow[dr, "\text{definition.}", "\mathbb{E}_{v,Y}"'] & K_S(T^*X) \arrow[r] \arrow[d, "\mathbb{E}_v"] \arrow[rr, bend left] & K(T^*X) \arrow[r] \arrow[d, "(\pi^*)^{-1}"] \arrow[dr, phantom, "(1)\ \circlearrowleft"] & K(T^*X\setminus S) \arrow[d, "v^*"] \\ & K_Y(X) \arrow[r] \arrow[rr, bend right] & K(X) \arrow[r] & K(X\setminus Y) \end{tikzcd}$$

localized sequence.

<u>proof</u> (1) Commutes since $\pi \circ v = Id$.

Existence of $K_S(T^*X) \xrightarrow{\mathbb{E}_v} K_Y(X)$ are due to the fact that the horizontals are homotopy cofiber sequences.

Note by $\mathbb{E}_{v,Y}$ the following composition

$$K_S(\mathcal{D}_X) \xrightarrow[gr_S]{} K_S(T^*X) \longrightarrow K_Y(X).$$

<u>Global product formula $[R\Gamma_{dr}(X, M)] = [R\Gamma(\mathbb{E}_{v,Y}(M))]$ as homotopy points of $K(k)$</u>

Suppose $X \xrightarrow{f} Z$ is a proper map between smooth varieties.

We have pushforward maps $Rf_* : D^b_{perf}(\mathcal{D}_X) \longrightarrow D^b_{perf}(\mathcal{D}_Z)$.

$$Rf_* : D^b_{perf}(X) \longrightarrow D^b_{perf}(Z)$$

If X is projective over k, then we get

$$R\Gamma_{dr} : D^b_{perf}(\mathcal{D}_X) \longrightarrow D^b_{perf}(k)$$

$$R\Gamma : D^b_{perf}(X) \longrightarrow D^b_{perf}(k)$$

Lemma 2.2 Let X be a smooth projective variety over k. then we have a homotopy commutative diagram

$$\begin{tikzcd} K(\mathcal{D}_X) \arrow[r, "gr"] \arrow[d, "R\Gamma_{dr}"'] & K(T^*X) \arrow[d, "(\pi^*)^{-1}"] \\ K(k) & K(X) \arrow[l, "R\Gamma"] \end{tikzcd}$$

Proof Note that $K(\mathcal{D}_X) \xrightarrow{gr} K(T^*X)$. Thus we only need to show the commutativity of

$$\begin{tikzcd} & K(\mathcal{D}_X) \arrow[r, "gr"] & K(T^*X) \\ {[\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}]} & & K(X) \arrow[ul, "w.e"'] \arrow[u, "\pi^*"'] \\ & \mathcal{F} \arrow[ul, mapsto] & \end{tikzcd}$$

$$\begin{tikzcd} K(\mathcal{D}_X) \arrow[d, "R\Gamma_{dr}"'] & K(X) \arrow[l] \arrow[dl, "R\Gamma"] \\ K(k) & \end{tikzcd}$$

But this follows from the standard fact of the theory of $\mathcal{D}_X$-modules. ⊠

Corollary 2.3 (Global product formula)

The homotopy points $[R\Gamma_{dr}(X,\mathcal{M})]$ and $[R\Gamma(\mathbb{E}_{v,Y}(\mathcal{M}))]$ are canonically identified.

proof

$$\begin{tikzcd} & K(\mathcal{D}_X) \arrow[r, "gr"] & K(T^*X) \arrow[r] & K(X) \arrow[r, "R\Gamma"] & K(k) \\ \mathcal{M} & K_S(\mathcal{D}_X) \arrow[u] \arrow[r, "gr_S"] \arrow[rr, bend right, "\mathbb{E}_{v,Y}"'] & K_S(T^*X) \arrow[u] \arrow[r, "\mathbb{E}_v"] & K_Y(X) \arrow[u] \arrow[ur] & \end{tikzcd}$$

usual way: $\mathcal{M} \mapsto [R\Gamma_{dr}(X,\mathcal{M})]$ by Lemma 2.2

micro-local way: $\mathcal{M} \mapsto [R\Gamma(\mathbb{E}_{v,Y}(\mathcal{M}))]$

Notation $\mathcal{E}_{v,Y}(\mathcal{M}) :== \det R\Gamma(\mathbb{E}_{v,Y}(\mathcal{M})) \in Pic^{\mathbb{Z}}(k)$.

Passing to determinant, Corollary 2.3 gives a canonical isomorphism

$$\eta_{dr,v,Y} : \det R\Gamma(X,\mathcal{M}) \longrightarrow \mathcal{E}_{v,Y}(\mathcal{M}).$$

Note that the epsilon factor $\varepsilon_{v,Y}(\mathcal{M})$ has a local nature in the sense that it only depends on the values of the 1-form v and $\mathcal{M}$ on an open neigh. of Y.

$v : X \setminus Y \longrightarrow T^*X \setminus S$.

<u>Lemma 2.4</u> Let $\mathcal{G}, \mathcal{F} \in D^b_{perf}(\mathcal{D}_X)$, v as above such that $SS(\mathcal{G}) \subseteq S$, $SS(\mathcal{F}) \subseteq S$.

Let U be an open neigh. of Y such that $\mathcal{F}|_U = \mathcal{G}|_U$. ($U \subset X$.)

① then $\mathbb{E}_{v,Y}(\mathcal{F}) = \mathbb{E}_{v,Y}(\mathcal{G})$ are canonicaly identified.

② If $v = \mu$ on an open neigh. U of Y, then $\mathbb{E}_{v,Y}(\mathcal{F}) = \mathbb{E}_{v,\mu}(\mathcal{F})$ are canonically identified.

<u>proof of ①</u>

$S_U = T^*U \cap S$

$$\begin{array}{ccccc} K_S(\mathcal{D}_X) & \longrightarrow & K_S(T^*X) & \longrightarrow & K_Y(X) \\ \downarrow & & \downarrow & & \| \\ K_{S_U}(\mathcal{D}_U) & \longrightarrow & K_{S_U}(T^*U) & \longrightarrow & K_Y(X) = K_Y(U). \end{array}$$

□

<u>Remark</u> ⓐ If X is smooth, then $K(X)$ equals ~~the~~ $G(X)$ the K-theory of the category of coherent sheaves on X. Thus the fiber $K_Y(X)$ of $K(X) \longrightarrow K(X \setminus Y)$ can be identified with $G(Y)$.

ⓑ If $Y = \coprod Y_i$, then $K_Y(X) = \prod K_{Y_i}(X)$ and

$$\mathbb{E}_{v,Y}(\mathcal{F}) = \sum_i \mathbb{E}_{v,Y_i}(\mathcal{F}) \quad \underline{\text{homotopy sum}}$$

and $\varepsilon_{v,Y}(\mathcal{F}) = \bigotimes_i \varepsilon_{v,Y_i}(\mathcal{F})$.

<u>Dubson-Kashiwara's index formula for $\mathcal{D}_X$-modules</u>

$\mathcal{M} \in D^b_{hol}(\mathcal{D}_X)$, $CC(\mathcal{M}) \in CH^d(T^*X)$ characteristic cycle.

<u>Index formula</u> $\chi(X, \mathcal{M}) \overset{X:\text{proj}}{=\!=\!=} \deg(CC(\mathcal{M}), T^*_X X)_{T^*X}$.

One can lift this formula to the level of K-theory spectra.

K-theory of $\mathcal{D}_X$-modules with good filtration

$$\begin{tikzcd}
KF_{S,0}(\mathcal{D}_X) \arrow[r, "gr_S"] \arrow[d, "\omega"', "\text{Forget } F"] & K_{S,0}(T^*X) \arrow[r] & K_0(T^*X) \arrow[r, "\text{Riemann-Roch morphism}"] \arrow[dd] & CH^*(T^*X) \arrow[d, "\text{zero-section}^!"] \\
K_{S,0}(\mathcal{D}_X) \arrow[r] \arrow[ur, "gr_S"] & K_0(\mathcal{D}_X) \arrow[ur] & & CH^d(X) \arrow[d, "\deg \text{ if } X \text{ proj}"] \\
 & & K_0(k) \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & CH^0(k)=\mathbb{Z}
\end{tikzcd}$$

For an object $(\mathcal{M}, F) \in$ ~~$KF_{S,0}(\mathcal{D}_X)$~~ $D^b_{hol}(\mathcal{D}_X)$ with good Filtration F.

Its image in $K_0(k)$ is $R\Gamma(X,\mathcal{M})$, and image in $CH^0(k)$ is $\chi(X,\mathcal{M})$.

Its image in $CH^d(T^*X)$ is $CC(\mathcal{M})$.

Commutativity means $\chi(X,\mathcal{M}) = \deg(CC\mathcal{M}, T^*_X X)_{T^*X}$. $\boxtimes$

§3 Universal properties of epsilon factors

Let $f: X \longrightarrow Y$ be a smooth morphism of smooth varieties over k.

Then one has pull-back functors $Lf^*: D^b_{perf}(\mathcal{D}_Y) \longrightarrow D^b_{perf}(\mathcal{D}_X)$.

Moreover for any $\mathcal{M} \in D^b_{perf}(\mathcal{D}_Y)$, we have

$$f^\circ SS(\mathcal{M}) := df(pr_2^{-1}(SS\mathcal{M})) = SS(Lf^*\mathcal{M})$$

where
$$\begin{tikzcd}
T^*X & T^*Y \times_Y X \arrow[l, "df"'] \arrow[r, "pr_1"] \arrow[d, "pr_2"] & X \arrow[d, "f"] \\
 & T^*Y \arrow[r, "\pi_Y"] & Y
\end{tikzcd}$$

Proposition 3.1 Assume that f is étale with follow diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X \setminus X' \arrow[l, hook'] \arrow[r, "f^*v"] \arrow[d] & T^*X \setminus f^\circ S \\
Y' \arrow[r, hook, "\text{closed}"'] & Y & Y \setminus Y' \arrow[l, hook'] \arrow[r, "v"] & T^*Y \setminus S
\end{tikzcd}$$

One has a commutative diagram of spectra:

$$\begin{tikzcd}
K_S(\mathcal{D}_Y) \arrow[r, "\mathbb{E}_{v,Y'}"] \arrow[d, "Lf^*"] & K_{Y'}(Y) \arrow[d, "f^*"] \arrow[drr, dashed] & & \\
K_{f^\circ S}(\mathcal{D}_X) \arrow[r, "\mathbb{E}_{v,X'}"] & K_{X'}(X) \arrow[rr, dashed, "\text{if } X \text{ proj}"'] & & K(k)
\end{tikzcd}$$

In particular if X and Y proj, and if $\mathcal{M} \in D^b_{perf}(\mathcal{D}_X)$ then two homotopy point

<u>proof</u> f^* exact. The diagram in question factor as

$$\begin{tikzcd}
K_S(\mathcal{D}_Y) \arrow[r] \arrow[d, "Lf^*"] & K_S(T^*Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & K_{Y'}(Y) \arrow[d, "f^*"] \\
K_{f\circ S}(\mathcal{D}_X) \arrow[r] & K_{f\circ S}(T^*X) \arrow[r] & K_{X'}(X)
\end{tikzcd}$$

given by

$$\begin{tikzcd}
T^*Y & T^*Y\times_Y X \arrow[l] \arrow[d, "\cong"] \\
& T^*X.
\end{tikzcd}$$

For the left square, it is enough to show the following square commutes

$$\begin{tikzcd}
KF_S(\mathcal{D}_Y) \arrow[r] \arrow[d] & K_S(\mathrm{gr}\,\mathcal{D}_Y) \arrow[d] \\
KF_{f\circ S}(\mathcal{D}_X) \arrow[r] & K_{f\circ S}(\mathrm{gr}\,\mathcal{D}_X)
\end{tikzcd}$$

defined by pullback along f and push-forward along the isom $f^*\mathrm{gr}\,\mathcal{D}_Y \simeq \mathrm{gr}\,\mathcal{D}_X$.

The diagram commutes since by construction of the filtered pull-back, we have $\mathrm{gr}(f^*(\mathcal{M},\mathcal{F})) \cong f^*(\mathrm{gr}_{\mathcal{F}}(\mathcal{M}))$ $\boxtimes$

Now we study push-forward by finite étale maps.

<u>Laumon</u> For any proper morphism $f: X \longrightarrow Y$ between smooth schemes, there is a commutative diagram of spectra

$$\begin{tikzcd}
KF(\mathcal{D}_X) \arrow[r, "\int_f"] \arrow[d, "\mathrm{gr}"] & KF(\mathcal{D}_Y) \arrow[d, "\mathrm{gr}"] \\
K(T^*X) \arrow[r, "G"] & K(T^*Y)
\end{tikzcd}$$

重视起来!

where G is induced by sending M to $R\,pr_{2*}\, df^! M[d_f]$, where d_f is the relative dimension of f.

Microlocal Riemann-Roch

Let $S \subseteq T^*X$ and $f_\circ S := pr(df^{-1}(S)) \subseteq T^*Y$, we have

$$\begin{tikzcd} KF_S(\mathcal{D}_X) \arrow[r, "S_f"] \arrow[d, "gr"] & KF_{f_\circ S}(\mathcal{D}_Y) \cancel{= FF} \arrow[d, "gr"] \\ K_S(T^*X) \arrow[r, "G"] & K_{f_\circ S}(T^*Y) \end{tikzcd}$$

Corollary Let $f: X \longrightarrow Y$ be a proper <u>étale</u> morphism with $S \subseteq T^*X$. v: non-vanishing 1-form on $Y \setminus Y'$ with $v(Y\setminus Y') \cap f_\circ S = \Phi$.

Since f is étale, $(f^*v)(X \setminus X') \cap S = \phi$.

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "P"] & X \arrow[d, "f"] \\ Y' \arrow[r, hook] & Y \end{tikzcd}$$

We have a commutative diagram

$$\begin{tikzcd} K_S(\mathcal{D}_X) \arrow[r] \arrow[d, "\mathbb{E}_{f^*v, X'}"'] & K_{S'}(\mathcal{D}_Y) \arrow[d, "\mathbb{E}_{v,Y}"] \\ K_{X'}(X) \arrow[r] & K_{Y'}(Y) \end{tikzcd}$$

Enough to check

$$\begin{tikzcd} K_S(T^*X) \arrow[r, "G"] \arrow[d] & K_{S'}(T^*Y) \arrow[d] \\ K_{X'}(X) \arrow[r, "Rf_*"] & K_{Y'}(Y) \end{tikzcd}$$

, then use localized sequence.

Filtered $\mathcal{D}_X$-modules

$\mathcal{D}_0 = \mathcal{O}_X \subseteq \mathcal{D}_1 \subseteq \cdots \subseteq \mathcal{D}_X$, $\mathcal{D}_i \mathcal{D}_j \subseteq \mathcal{D}_{i+j}$ and $\bigcup_i \mathcal{D}_i = \mathcal{D}_X$.

Each $\mathcal{D}_i$ is a locally free $\mathcal{O}_X$-module.

A filtered $\mathcal{D}_X$-module consists of a pair $(\mathcal{M}, \mathcal{F}_\bullet)$, where $\mathcal{M}$ is a $\mathcal{D}_X$-module and $\mathcal{F}_\bullet$ is an increasing $\mathbb{Z}$-filtration of $\mathcal{M}$ by $\mathcal{O}_X$-submodules such that $\mathcal{F}_i = 0$ for $i \ll 0$, $\bigcup_i \mathcal{F}_i = \mathcal{M}$ and $\mathcal{D}_i \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$.

A filtered $\mathcal{D}_X$-module is quasi-coherent if each $\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_X$-module.

§4 ε-factor supported on points

<u>Idea</u> Patel's epsilon factor $K_S(\mathcal{D}_X) \xrightarrow{\mathbb{E}_{\nu,Y}} K_Y(X)$ is expected to supported on a closed subset Y of X of codimension 1.
Using several 1-forms at once, one can replace Y by a smaller closed subset Z, even of codimension $d = \dim X$.

This observation is due to <u>Michael Groechenig</u>

<u>Notation</u> I finite non-empty ordered set.

$$\Delta^I := \frac{\mathbb{A}^I}{\langle \Sigma \lambda_i - 1 \rangle}$$ affine space of <u>dim $|I|-1$</u>, defined by $\Sigma \lambda_i = 1$.

— Let X be a smooth d-dimensional scheme over a field k of characteristic 0.
$\underset{\text{closed}}{Z \subseteq X} \hookleftarrow U = X \setminus Z$. $S \subseteq T^*X$ conical closed.

— Consider an open covering $U = \bigcup_{i=1}^{m} U_i$ and for each $i = 1, \dots, m$, a non-where vanishing 1-form $\nu_i \in \Omega^1_X(U_i)$ such that for each ordered subset $I = \{i_1 < \dots < i_\ell\} \subseteq \{1, \dots, m\}$, we have for $U_{i_1, \dots, i_\ell} = \bigcap_{j=1}^{\ell} U_{i_j}$ that the image of the morphism

$$\nu_\Delta : U_{i_1, \dots, i_\ell} \times \Delta^I \longrightarrow T^*X$$
$$\nu_\Delta(\lambda_1, \dots, \lambda_\ell) = \sum_{j=1}^{\ell} \lambda_j \nu_{i_j}.$$

such that $\mathrm{Im}(\nu_\Delta) \cap S = \emptyset$.

<u>Lemma + Definition</u> There exists a morphism
$$(\nu)^* : K(T^* X \setminus S) \longrightarrow K(X \setminus Z)$$
such that for all $i = 1, \dots, m$, we have a commutative diagram

$$\begin{tikzcd}
K_S(\mathcal{S}_X) \arrow[r] \arrow[dr, "\det \mathbb{E}_{v,Z}"'] \arrow[dddd, "\varepsilon_{v,Z} \det"'] & K_S(T^*X) \arrow[r] \arrow[d, "\phi_{\underline{v}}"] & K(T^*X) \arrow[r] \arrow[d, "(\pi^*)^{-1}"] & K(T^*X \setminus S) \arrow[r, "v_i^*"] \arrow[d, "\underline{v}^*"] & K(U_i) \arrow[dl] \\
& K_Z(X) \arrow[r] \arrow[d, "\text{if } Z \text{ proper}"] & K(X) \arrow[r] & K(X\setminus Z) \arrow[d, phantom, "\shortparallel"] & \\
& K(Z) \arrow[ddl, "\det"] & & K(U) & \\
& & & & \\
\mathrm{Pic}^{\mathbb{Z}}(k) & & & &
\end{tikzcd}$$

重要

Furthermore, we have $\underline{v}^* \circ \pi^* \simeq \mathrm{Id}$.

<u>Sketch of the proof for $m=2$</u>

The idea is to show that on $U_{12} = U_1 \cap U_2$, one can construct a linear homotopy $v_t : U_{12} \times \mathbb{A}^1 \longrightarrow T^*U_{12}$ between the sections $v_1|_{U_{12}}$ and $v_2|_{U_{12}}$ by $v_t = (1-t)v_1 + t v_2$ for $t \in \mathbb{A}^1$. In this step, we use that v_1 and v_2 are linearly independent on U_{12}.

<u>Claim</u> there is a commutative diagram of spectra

$$\begin{tikzcd}
K(T^*X) \arrow[r, "v_1^*"] \arrow[d, "v_2^*"] & K(U_1) \arrow[d] \\
K(U_2) \arrow[r] & K(U_{12})
\end{tikzcd}$$

By Thomason - Trobaugh, alg. K-theory satisfies Zariski descent, the system of coherent homotopies alluded to above yields the descent datum for a morphism $K(T^*X\setminus S) \longrightarrow K(U)$.

$\boxed{\text{proof of claim}}$ We show $(v_1|_{U_{12}})^* = (v_2|_{U_{12}})^*$.

$$\begin{tikzcd}
U_{12}\times\{0\} \arrow[rr, "v_0"] \arrow[dr, "i_0"] & & T^*X \\
& U_{12}\times\mathbb{A}^1 \arrow[ur, "v_t"] & \\
U_{12}\times\{1\} \arrow[ur, "i_1"] \arrow[uurr, bend right, "v_1"'] & &
\end{tikzcd}$$

Since the K-theory of regular schemes is $\mathbb{A}^1$-invariant, and i_0, i_1 are sections of $\pi : U_{12}\times\mathbb{A}^1 \longrightarrow U_{12}$, we have that the pullback along i_0 and i_1 are inverse to π^*, therefore $v_0^* \simeq i_0^* \circ v_t^* \simeq (\pi^*)^{-1} \circ v_t^* \simeq i_1^* \circ v_t^* \simeq v_1^*$ $\boxtimes$

Remark If $\dim Z = 0$ then $\mathcal{E}_v(M) = \bigotimes \mathcal{E}_{v,z}(M)$

$$\begin{tikzcd}
K_S(\mathcal{D}_X) \arrow[r] \arrow[dr, "\det", "\mathbb{E}_{v,Z}"'] \arrow[ddd, "\mathcal{E}_{v,Z}"', "\det"] & K_S(T^*X) \arrow[r] \arrow[d, "\phi_{\underline{v}}"] & K(T^*X) \arrow[r] \arrow[d, "(\pi^*)^{-1}"] & K(T^*X \setminus S) \arrow[r, "v_i^*"] \arrow[d, "\underline{v}^*"] & K(U_i) \arrow[dl] \\
 & K_Z(X) \arrow[r] \arrow[dl, "\text{if } Z \text{ proper}"] & K(X) \arrow[r] & K(X \setminus Z) \arrow[d, equal] & \\
K(k) \arrow[d, "\det"] & & & K(U) & \\
\mathrm{Pic}^{\mathbb{Z}}(k) & & & &
\end{tikzcd}$$

重要

Furthermore, we have $\underline{v}^* \circ \pi^* \simeq \mathrm{Id}$.

<u>Sketch of the proof for $m=2$</u>

The idea is to show that on $U_{12} = U_1 \cap U_2$, one can construct a linear homotopy $v_t : U_{12} \times \mathbb{A}^1 \longrightarrow T^*U_{12}$ between the sections $v_1|_{U_{12}}$ and $v_2|_{U_{12}}$ by $v_t = (1-t)v_1 + t v_2$ for $t \in \mathbb{A}^1$. In this step, we use that v_1 and v_2 are linearly independent on U_{12}.

<u>Claim</u> there is a commutative diagram of spectra

$$\begin{tikzcd}
K(T^*X) \arrow[r, "v_1^*"] \arrow[d, "v_2^*"] & K(U_1) \arrow[d] \\
K(U_2) \arrow[r] & K(U_{12})
\end{tikzcd}$$

By Thomason-Trobaugh, alg. K-theory satisfies Zariski descent, the system of coherent homotopies alluded to above yields the descent datum for a morphism $K(T^*X \setminus S) \longrightarrow K(U)$.

proof of claim We show $(v_1|_{U_{12}})^* = (v_2|_{U_{12}})^*$.

$$\begin{tikzcd}
U_{12} \times \{0\} \arrow[rr, "v_0"] \arrow[dr, "i_0"'] & & T^*X \\
 & U_{12} \times \mathbb{A}^1 \arrow[ur, "v_t"'] & \\
U_{12} \times \{1\} \arrow[ur, "i_1"] \arrow[uurr, bend right, "v_1"'] & &
\end{tikzcd}$$

Since the K-theory of regular schemes is $\mathbb{A}^1$-invariant, and i_0, i_1 are sections of $\pi : U_{12} \times \mathbb{A}^1 \longrightarrow U_{12}$, we have that the pullback along i_0 and i_1 are inverse to π^*,

therefore $v_0^* \simeq i_0^* \circ v_t^* \simeq (\pi^*)^{-1} \circ v_t^* \simeq i_1^* \circ v_t^* \simeq v_1^*$ $\boxtimes$

Remark If $\dim Z = 0$, then $\mathcal{E}_{\underline{v},Z}(M) = \bigotimes_{z \in \pi_0(Z)} \mathcal{E}_{\underline{v},z}(M)$.

§5 Twist formula

X projective smooth.

$\mathcal{F} \in D^b_{hol}(\mathcal{D}_X)$.

$\mathcal{G} \in D^b_{hol}(\mathcal{D}_X)$ with $SS(\mathcal{G}) \subseteq T^*_X X \Rightarrow \mathcal{G}$ vector bundle with connection.

We show $\quad \mathcal{E}_{dR}(X, \mathcal{G} \otimes \mathcal{F}) \simeq \langle \det \mathcal{G}, CC\mathcal{F} \rangle$

We first recall Levine's homotopy coniveau tower

X/k smooth. $K(X)$ = K-theory spectrum of coherent sheaves on X.

$\Delta^n = \mathrm{Spec} \dfrac{k[t_0, \dots, t_n]}{(\Sigma t_i - 1)}$ usual n-simplex.

A face of Δ^n is a closed subscheme defined by equations of the form
$$t_{i_1} = \dots = t_{i_s} = 0.$$
Then one defines $K^{(q)}(X, p) :== \underset{\overrightarrow{W}}{\mathrm{holim}}\, K_W(X \times \Delta^p)$ where the homotopy limit is taken over closed subschemes $W \subseteq X \times \Delta^p$ such that $\mathrm{codim}_{X\times F}(W \cap X \times F) \geq q$ for all faces $F \subseteq \Delta^p$.

We put $K^{(q)}(X) :== K^{(q)}(X, 0) = \underset{\substack{\overrightarrow{W} \\ \mathrm{codim}_X W \geq q}}{\mathrm{holim}}\, K_W(X)$.

The spectra $K^{(q)}(X, p)$ form a ~~sp~~ simplicial spectrum, and we let $K^{(q)}(X, -)$ denote the corresponding total spectrum.

Moreover, one have a tower of spectra (homotopy coniveau tower)
$$\cdots \to K^{(q)}(X, -) \longrightarrow K^{(q-1)}(X, -) \longrightarrow \cdots \longrightarrow K^{(0)}(X, -)$$
with augmentation maps $\eta_q : K^{(q)}(X) \longrightarrow K^{(q)}(X, -)$.

with the followng properties proved by Levine:

(1) pull-back by smooth map

(2) the composition $K(X) \longrightarrow K^{(0)}(X) \longrightarrow K^{(0)}(X, -)$ is a weak equivalence.

(3) The cofibers $K^{(p/p+1)}(X, -)$ of the homotopy coniveau tower are naturally weak equivalence to Bloch's higher Chow group cycle complex, In particular, there is a

functorial isomorphism

$$CH^d(X) \longrightarrow \pi_0\big(K^{(d)}(X,-)\big) \quad \text{if } d=\dim X.$$

(4) tensor product induces natural pairings

$$K^{(d)}(X,-)\wedge K^{(d')}(X,-)\longrightarrow K^{(d+d')}(X,-).$$

In particular $K(X)\wedge K^{(d)}(X,-)\longrightarrow K^{(d)}(X,-)$.

<u>Now we always assume that X is smooth projective over k</u>

Patel's result $gr_S : K_S(\mathcal{D}_X)\longrightarrow K_S(T^*X)$ $\quad S\subseteq T^*X$

take limit, we get $K_{hol}(\mathcal{D}_X)\longrightarrow K^{(d)}(T^*X)$.

For a holonomic $\mathcal{D}_X$-module $\mathcal{F}$, we set $\mathcal{E}_{dR}(X,\mathcal{F}) = \det R\Gamma_{dR}(X,\mathcal{F})\in \mathrm{Pic}^{\mathbb{Z}}(k)$

<u>Consider</u> $CC^K : K_{hol}(\mathcal{D}_X)\xrightarrow{\ \varepsilon\ } K^{(d)}(T^*X)\longrightarrow K^{(d)}(T^*X,-)$

<u>Twisted pullback by σ_X</u> : $\sigma_X : X\longrightarrow T^*X$ zero section

$$\begin{tikzcd}
K(T^*X) \arrow[rr, "\sigma^+"] \arrow[dr, "\sigma^*"'] & & K(X) \\
& K(X) \arrow[ur, "-\otimes\omega_X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
K^{(d)}(T^*X,-) \arrow[rr, "\sigma^+"] \arrow[dr, "\sigma^*"'] & & K^{(d)}(X,-) \\
& K^{(d)}(X,-) \arrow[ur, "-\otimes\omega_X"'] &
\end{tikzcd}$$

canonical line bundle

We set $CC := \sigma^+\circ CC^K$

let $F^\flat : K_X(\mathcal{D}_X)\longrightarrow K(X)$ forgetting the $\mathcal{D}_X$-module structure

$SS\subseteq T^*_X X$

<u>regarded $\mathcal{O}_X$-module</u>

<u>shifted</u>

<u>Recall that</u>

$$\begin{tikzcd}
K(\mathcal{D}_X) \arrow[r, "R\Gamma_{dR}"] \arrow[d, "gr"'] & K(k) \\
K(T^*X) \arrow[r, "\sigma^+"] & K(X) \arrow[u, "f_*"']
\end{tikzcd}$$

<u>Theorem (Abe-Patel)</u> The following diagram commutes up to homotopy equivalence

$$\begin{tikzcd}
K_X(\mathcal{D}_X) \wedge K_{hol}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "F^{\nabla} \wedge CC"'] & K_{hol}(\mathcal{D}_X) \arrow[r, "gr"] \arrow[d, "R\Gamma_{dR}"] \arrow[dr, phantom, "\circlearrowright"] & K(T^*X) \arrow[d, "\sigma^+"] \\
K(X) \wedge K^{(d)}(X,-) \arrow[r, "{\langle -,- \rangle^K_{(d,-)}}"] \arrow[d, "\otimes"] \arrow[rr, bend left] \arrow[dr, phantom, "\text{definition}"] & K(k) & K(X) \arrow[l] \arrow[dl, equal] \\
K^{(d)}(X,-) \arrow[r] & K^{(0)}(X,0) \cong K(X) \arrow[u, "f_*"'] &
\end{tikzcd}$$

$\uparrow$
<u>coniveau tower</u>

<u>proof</u> We only need to show A is commutative.

Since σ^+ commutes with $\otimes$ and argumentation, we reduce to

$$\begin{tikzcd}
 & K_X(\mathcal{D}_X) \wedge K_{hol}(\mathcal{D}_X) \arrow[rr, "\otimes"] \arrow[dl, "\pi^* \wedge \varepsilon"'] \arrow[dd] & & K(\mathcal{D}_X) \arrow[dl, "gr"'] \arrow[dd, "\sigma^* \circ gr"] \\
K(T^*X) \wedge K^{(d)}(T^*X) \arrow[rr, dashed, "F^{\nabla} \wedge CC^K"] \arrow[dr, "\sigma^* \wedge Id"'] & & K(T^*X) \arrow[dr, "\sigma^*"] & \\
 & K(X) \wedge K^{(d)}(X,-) \arrow[rr, "\otimes"'] & & K(X)
\end{tikzcd}$$

As F^{∇} is homotopic to $\sigma^* \circ \pi^*$, we are reduced to show that the following diagram commutes

$$\begin{tikzcd}
K_X(\mathcal{D}_X) \wedge K_{hol}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "\pi^* \wedge \varepsilon"'] & K(\mathcal{D}_X) \arrow[d, "gr"] \\
K(T^*X) \wedge K^{(d)}(T^*X) \arrow[r, "\otimes"] & K(T^*X)
\end{tikzcd}$$

By def, this diagram factors as

$$\begin{tikzcd}
K_X(\mathcal{D}_X) \wedge K_{hol}(\mathcal{D}_X) \arrow[r] \arrow[d, "\pi^* \wedge \varepsilon"] \arrow[dr, phantom, "\circlearrowright"] & K_X(\mathcal{D}_X) \wedge K(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "\pi^* \wedge gr"] \arrow[dr, phantom, "\circlearrowright"] & K(\mathcal{D}_X) \arrow[d, "gr"] \\
K(T^*X) \wedge K^{(d)}(T^*X) \arrow[r] & K(T^*X) \wedge K(T^*X) \arrow[r, "\otimes"] & K(T^*X)
\end{tikzcd}$$

Corollary (twist formula)

— $\mathcal{F}$ holonomic $\mathcal{D}_X$-module on projective smooth scheme X

$\mathcal{G}$ vector bundle with connection. then forgetting the connection, $\mathcal{G}$ give rises to a natural object $\det\mathcal{G} \in \pi K(X)$.

— $\mathcal{F}$ gives rise to an object of the Picard groupoid associated to $K_{hol}(\mathcal{D}_X)$, hence an object of $\pi K^{(d)}(X,-)$ via the morphism $CC : K_{hol}(\mathcal{D}_X) \longrightarrow K^{(d)}(X,-)$.

We denote the corresponding object by $CC\mathcal{F} \in \pi K^{(d)}(X,-)$.

— Applying the pairing $\langle -,- \rangle_{(d)} : \pi K(X) \wedge \pi K^{(d)}(X,-) \longrightarrow \mathrm{Pic}^{\mathbb{Z}}(k)$

we get $\langle \det\mathcal{G}, CC\mathcal{F} \rangle \in \mathrm{Pic}^{\mathbb{Z}}(k)$.

Corollary One has a natural isomorphism in $\mathrm{Pic}^{\mathbb{Z}}(k)$

$$\mathcal{E}_{dR}(X, \mathcal{G}\otimes\mathcal{F}) \cong \langle \det\mathcal{G}, CC\mathcal{F} \rangle$$

proof

$$\begin{tikzcd}
\pi K_X(\mathcal{D}_X) \wedge \pi K_{hol}(\mathcal{D}_X) \arrow[r] \arrow[d, "F \wedge CC"'] \arrow[dr, phantom, "\circlearrowleft \text{ previous thm.}"] \arrow[drr, bend right=60, "\text{give } \langle \det\mathcal{G},\, CC\mathcal{F} \rangle"'] & \pi K(\mathcal{D}_X) \arrow[d] \arrow[d, bend left=60, "\text{give } \mathcal{E}_{dR}(X,-)"] \\
\pi K(X) \wedge \pi K^{(d)}(X,-) \arrow[r] & \mathrm{Pic}^{\mathbb{Z}}(k)
\end{tikzcd}$$

Theorem $f \in \mathrm{End}(\mathcal{F})$, $g \in \mathrm{End}(\mathcal{G})$. $f\otimes g \in \mathrm{End}(R\Gamma_{dR}(X, \mathcal{F}\otimes\mathcal{G}))$

put $\mathcal{E}_{dR}(X, \mathcal{F}\otimes\mathcal{G}; f\otimes g) = \mathrm{Tr}(f\otimes g \mid R\Gamma_{dR}(X, \mathcal{F}\otimes\mathcal{G})) \in k^{\times}$.

$r_{\mathcal{G}}$ = generic rank of $\mathcal{G}$, i.e., image of $\mathcal{G}$ by the map $\pi_0(\mathrm{Pic}^{\mathbb{Z}}(X)) \longrightarrow \mathbb{Z}$.

Then

$$\mathcal{E}_{dR}(X, \mathcal{F}\otimes\mathcal{G}; f\otimes g) = \mathcal{E}_{dR}(X, \mathcal{F}; f)^{r_{\mathcal{G}}} \times \underline{\langle \det\mathcal{G}, CC\mathcal{F} \rangle}(g, \mathrm{Id})$$

Trace of $g\otimes\mathrm{Id} \in \mathrm{End}(\langle \det\mathcal{G}, CC\mathcal{F} \rangle)$