

de Rham epsilon factor

§1 Motivation

Beilinson's question on ϵ -factorization of motivic (ℓ -adic or de Rham) epsilon factors

Example k : finite field. X/k projective smooth curve.

\mathcal{F} : ℓ -adic sheaf on X

The global epsilon factor $\epsilon(\mathcal{F}, X) = \det(-\text{Frob}_k; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1}$

is the constant term of the functional equation for Grothendieck L -functions of \mathcal{F}

$$L(X, \mathcal{F}, t) = t^{-\chi(X_{\bar{k}}, \mathcal{F})} \cdot \epsilon(\mathcal{F}, X) \cdot L(X, D(\mathcal{F}), t^{-1})$$

||det

$$\det(1-t \text{Frob}_k; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1}$$

where $\chi(X_{\bar{k}}, \mathcal{F}) = \sum_i (-1)^i \dim H_{\text{ét}}^i(X_{\bar{k}}, \mathcal{F})$ is the étale Euler-Poincaré number.

$\text{Frob}_k \in \text{Gal}(\bar{k}/k)$ is the geometric Frobenius (inverse of Frob with $x \mapsto x^{\#k}$)

The global epsilon factor $\epsilon(X, \mathcal{F})$ satisfies the following product formula, conjectured by P. Deligne and proved by Laumon:

$$\epsilon(\mathcal{F}, X) = \left(\prod_{x \in |X|} \epsilon_x(\omega, \mathcal{F}) \right) \cdot q^{(1-g)\text{rank } \mathcal{F}}, \quad \psi: \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_c^{\times} \text{ fixed char.}$$

where ω is a non-vanishing meromorphic 1-form on X , $\omega \in \Omega_{k(X)}^1$.

$\epsilon_x(\omega, \mathcal{F})$ is the local epsilon factor, whose existence is suggested by Langlands program, proved by Deligne or Laumon using Local Fourier transform.

The local data $\epsilon_x(\omega, \mathcal{F})$ depends only on the restriction

$$(\omega, \mathcal{F}) \Big|_{\text{Spec } \widehat{\mathcal{O}_{X,x}}} \leftarrow \text{completion.}$$

Open Question

— Geometric ϵ -factorization of $\det R\Gamma(X, \mathcal{F})$, which gives Laumon's product formula after taking trace of Frobenius. (even open for curves)

— Higher dimensional analogues?

We have higher class field theory, but no higher Langlands program!

Today We present such a theory for de Rham E-factor due to Deepam Patel. Some rank 1 theory on curves are due to Beilinson-Bloch-Esnault. See also Deligne's IHES lectures.

Quick review

In this talk, k always be a field of characteristic 0.

And X is a smooth variety of dimension d over k .

~~choose d 's 1-form v_1, \dots, v_d~~

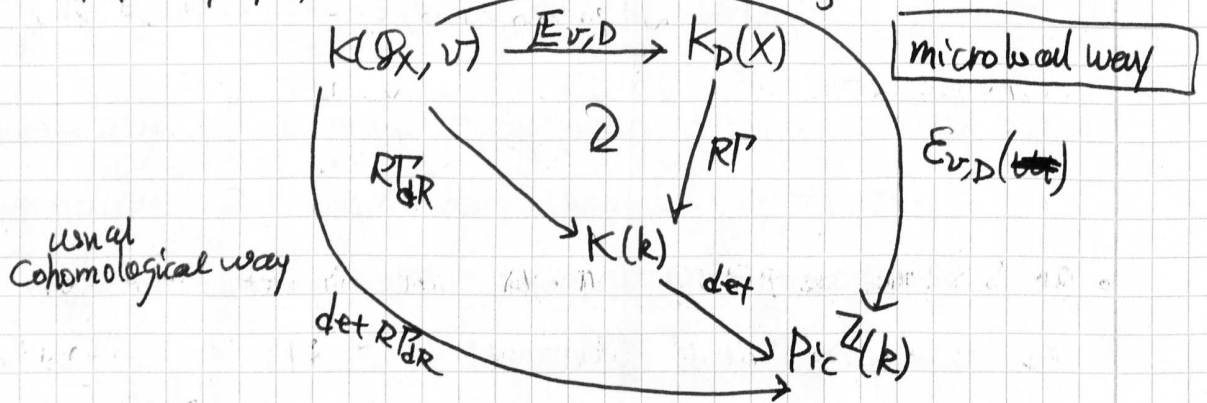
$U \subseteq X \xrightarrow{\text{open}} D = X \setminus U$. $v: U \rightarrow T^*X$ 1-form, $v \in \Omega_X^1(U)$.

Let $K(\mathcal{D}_X, v)$ be the K-theory spectrum of ~~locally finitely~~ perfect \mathcal{D}_X -modules \mathcal{M} on X such that $v(U) \cap \text{SS}(\mathcal{M}) = \emptyset$.

$K_D(X) = K$ -theory spectrum of coherent sheaves on X which are set theoretically supported on D .

Theorem (Patel) $\exists K(\mathcal{D}_X, v) \xrightarrow{E_{v,D}} K_D(X)$

If X is proper, we have a commutative diagram



Thus $\det RP_{dR}(X, \mathcal{M}) \cong E_{v,D}(\mathcal{M})$.

Moreover, we can choose d 's 1-form v_1, \dots, v_d such that D may choose to be of dimension 0.

Put $\pi = (v_1, \dots, v_d)$.

Then $\det RP_{dR}(\mathcal{M}) \cong E_{v,D}(\mathcal{M}) \cong \bigotimes_{x \in \pi_0(D)} E_{\pi, x}(\mathcal{M})$.

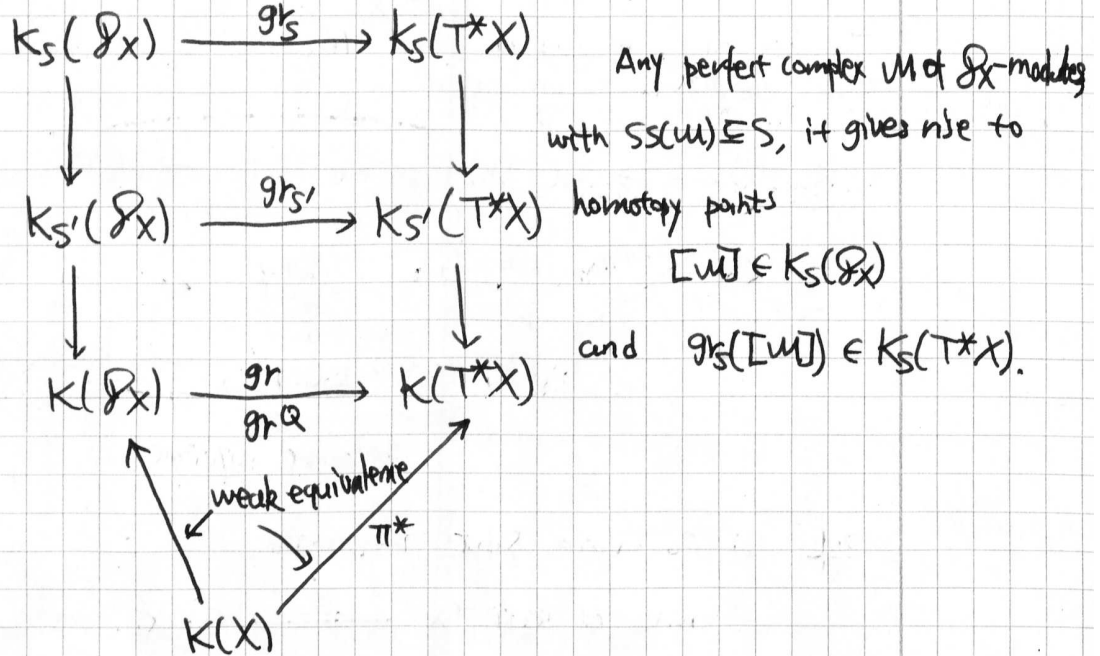
is the promised factorization formula in higher dimensions.

§ 2 Construction of epsilon factors

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Recall from previous ~~work~~ talks about the micro-local description of singular supports

$S' \subseteq S \subseteq T^*X$ conical closed subsets, we have a homotopy commutative diagram



where $K(X) \rightarrow K(T^*X)$ is defined by pull-back by $T^*X \xrightarrow{\pi} X$

$K(X) \rightarrow K(\mathcal{D}_X)$ is defined by $[F] \mapsto [\mathcal{D}_X \otimes_{\mathcal{O}_X} F]$.

Then invert the weak equivalences gives us a morphism

$$K(\mathcal{D}_X) \xrightarrow{\text{gr}^Q} K(T^*X).$$

- gr is defined as follows π affine and $\pi_* \mathcal{O}_{T^*X} \cong \mathcal{O}_X$.

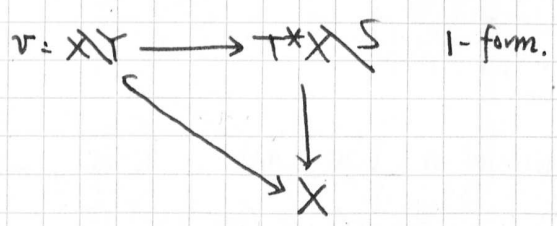
We have an equivalence of K-theory spectra $\pi_*: K(T^*X) \rightleftarrows K(\mathcal{O}_X)$

- The morphisms g_S and $g_{S'}$ are induced from gr

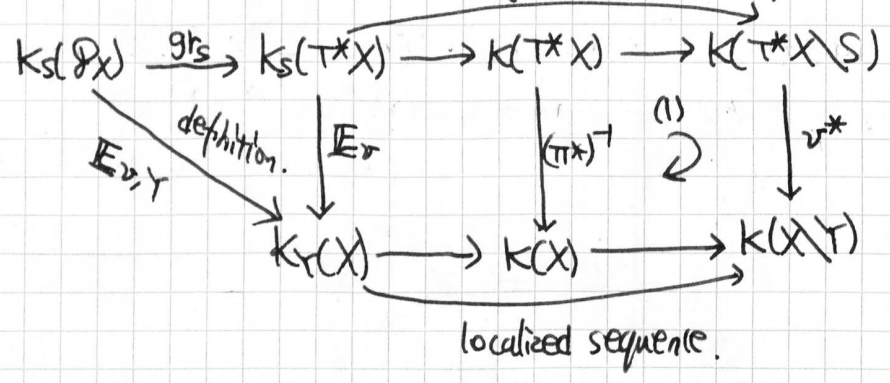


We can prove that The homotopy morphism gr^Q and gr are canonically identified.

Lemma 3.1 $Y \subseteq X$ closed subset
 $S \subseteq T^*X$ conical closed subset.



then one has a commutative diagram: localized sequence



proof (1) Commutes since $\pi \circ v = \text{Id}$.
 Existence of $K_S(T^*X) \xrightarrow{\mathbb{E}_v} K_Y(X)$ are due to the fact
 that the horizontals are homotopy cofiber sequences.

Note by $\mathbb{E}_{v,Y}$ the following composition

$$K_S(\mathcal{O}_X) \xrightarrow{g_S} K_S(T^*X) \longrightarrow K_Y(X).$$

Global product formula $[R\Gamma_{\text{dr}}(X, \mathcal{U})] = [R\Gamma(\mathbb{E}_{v,Y}(\mathcal{U}))]$ as homotopy points
 of $K(k)$

Suppose $X \xrightarrow{f} Z$ is a proper map between smooth varieties.

We have pushforward maps $Rf_*: D_{\text{perf}}^b(\mathcal{O}_X) \longrightarrow D_{\text{perf}}^b(\mathcal{O}_Z)$.

$$Rf_*: D_{\text{perf}}^b(X) \longrightarrow D_{\text{perf}}^b(Z)$$

If X is projective over k , then we get

$$R\Gamma_{\text{dr}}: D_{\text{perf}}^b(\mathcal{O}_X) \longrightarrow D_{\text{perf}}^b(k)$$

$$R\Gamma: D_{\text{perf}}^b(X) \longrightarrow D_{\text{perf}}^b(k)$$

Lemma 2.2 Let X be a smooth projective variety over k . Then

We have a homotopy commutative diagram

$$\begin{array}{ccc}
 K(\mathcal{O}_X) & \xrightarrow{g^*} & K(T^*X) \\
 \text{RT}_{\text{dir}} \downarrow & & \downarrow (\pi^*)^{-1} \\
 K(k) & \xleftarrow{R\Gamma} & K(X)
 \end{array}$$

Proof

Note that $K(\mathcal{O}_X) \xrightarrow{g^*} K(T^*X)$. Thus we only need to show the commutativity of

$$\begin{array}{ccc}
 [\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F}] & \xleftarrow{w.e} & K(X) \\
 \downarrow \mathcal{F} & & \uparrow \pi^* \\
 & & K(X)
 \end{array}$$

$$\begin{array}{ccc}
 K(\mathcal{O}_X) & \xleftarrow{\quad} & K(X) \\
 \text{RT}_{\text{dir}} \downarrow & & \swarrow R\Gamma \\
 K(k) & &
 \end{array}$$

But this follows from the standard fact of the theory of \mathcal{O}_X -modules. \square

Corollary 2.3 (Global product formula)

The homotopy points $[RT_{\text{dir}}(X, \mathcal{U})]$ and $[R\Gamma(E_{\nu, \gamma}(\mathcal{U}))]$ are canonically identified.

proof

$$\begin{array}{ccccccc}
 & & \text{usual way} & & & & \\
 & & \xrightarrow{\quad} & & & & [RT_{\text{dir}}(X, \mathcal{U})] \text{ by Lemma 2.2} \\
 & & & & & & \\
 K(\mathcal{O}_X) & \xrightarrow{g^*} & K(T^*X) & \rightarrow & K(X) & \xrightarrow{R\Gamma} & K(k) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \mathcal{U} & & & & & & [R\Gamma(E_{\nu, \gamma}(\mathcal{U}))] \\
 & & & & & & \\
 K_S(\mathcal{O}_X) & \xrightarrow{g_S^*} & K_S(T^*X) & \xrightarrow{E_{\nu}} & K_r(X) & & \\
 & & \searrow & \xrightarrow{E_{\nu, \gamma}} & & & \\
 & & & \text{micro-local way} & & &
 \end{array}$$

Notation $E_{\nu, \gamma}(\mathcal{U}) := \det R\Gamma(E_{\nu, \gamma}(\mathcal{U})) \in \text{Pic}^{\mathbb{Z}}(k)$.

Passy to determinant, Corollary 2.3 gives a canonical isomorphism

$$\eta_{\text{dir}, \nu, \gamma} : \det R\Gamma(X, \mathcal{U}) \longrightarrow E_{\nu, \gamma}(\mathcal{U}).$$

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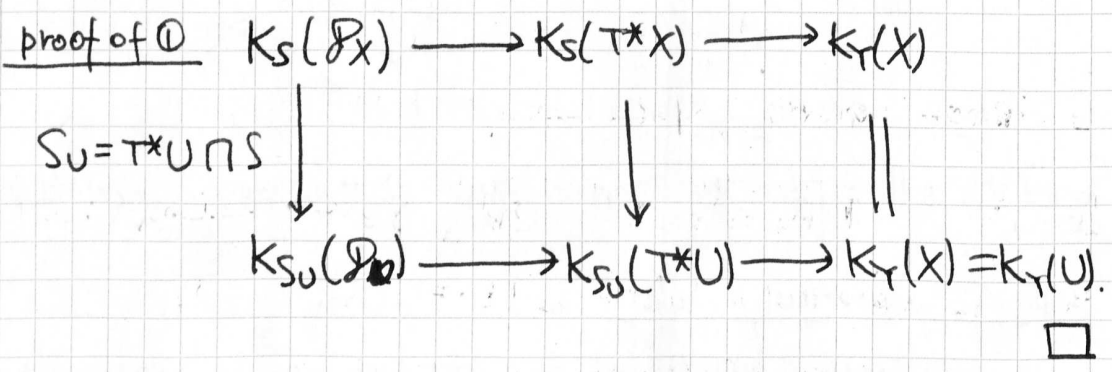
Note that the epsilon factor $E_{v,Y}(M)$ has a local nature in the sense that it only depends on the values of the 1-form v and M on an open neigh. of Y .

$$v: X \rightarrow T^*X/S$$

Lemma 2.4 Let $S, F \in D_{\text{hol}}^b(\mathcal{D}_X)$, v as above such that $SS(S) \subseteq S$, $SS(F) \subseteq S$.

Let U be an open neigh. of Y such that $F|_U = S|_U$.

- ① then $E_{v,Y}(F) = E_{v,Y}(S)$ are canonically identified.
- ② If $v = \mu$ on an open neigh. U of Y , then $E_{v,Y}(F) = E_{\mu,U}(F)$ are canonically identified.



Remark ② If X is smooth, then $K(X)$ equals $G(X)$ the K-theory of the category of coherent sheaves on X . Thus the fiber $K_Y(X)$ of $K(X) \rightarrow K(X/Y)$ can be identified with $G(Y)$.

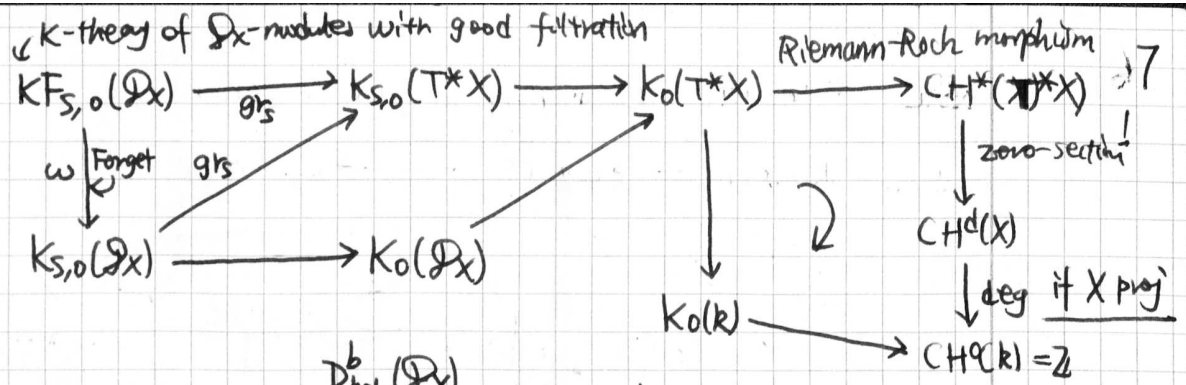
- ③ If $Y = \coprod Y_i$, then $K_Y(X) = \prod K_{Y_i}(X)$ and $E_{v,Y}(F) = \sum_i E_{v,Y_i}(F)$ homotopy sum and $E_{v,Y}(F) = \otimes_i E_{v,Y_i}(F)$.

Dubson-Kashiwara's index formula for \mathcal{D}_X -modules

$M \in D_{\text{hol}}^b(\mathcal{D}_X)$, $CC(M) \in CH^d(T^*X)$ characteristic cycle.

Index formula $\chi(X, M) \stackrel{X: \text{proj}}{=} \deg(CC(M), T^*X)_{T^*X}$.

One can lift this formula to the level of K-theory spectra.



For an object $(u, F) \in \text{KFs}_{s,0}(\mathcal{O}_X)$ with good filtration F .

Its image in $\text{K}_0(k)$ is $\text{rk}(X, u)$, and image in $\text{CH}^0(k)$ is $\chi(X, u)$.

Its image in $\text{CH}^d(T^*X)$ is $\text{CC}(u)$.

Commutativity means $\chi(X, u) = \text{deg}(\text{CC}(u), T^*X)_{T^*X}$. □

§3 Universal properties of epsilon factors

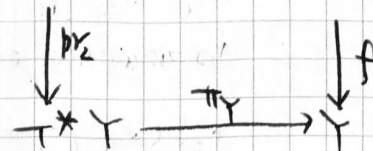
Let $f: X \rightarrow Y$ be a smooth morphism of smooth varieties over k .

Then one has pullback functors $Lf^*: D_{\text{perf}}^b(\mathcal{O}_Y) \rightarrow D_{\text{perf}}^b(\mathcal{O}_X)$.

Moreover for any $u \in D_{\text{perf}}^b(\mathcal{O}_Y)$, we have

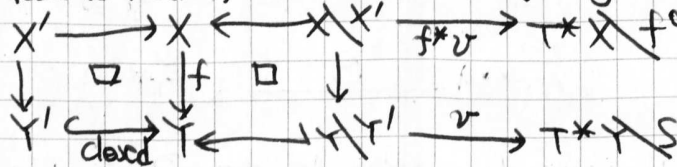
$$f^* \text{SS}(u) := \text{df}(\text{pr}_2^{-1}(\text{SS}u)) = \text{SS}(Lf^*u)$$

where $T^*X \xleftarrow{\text{df}} T^*Y \times_Y X \xrightarrow{\text{pr}_1} X$

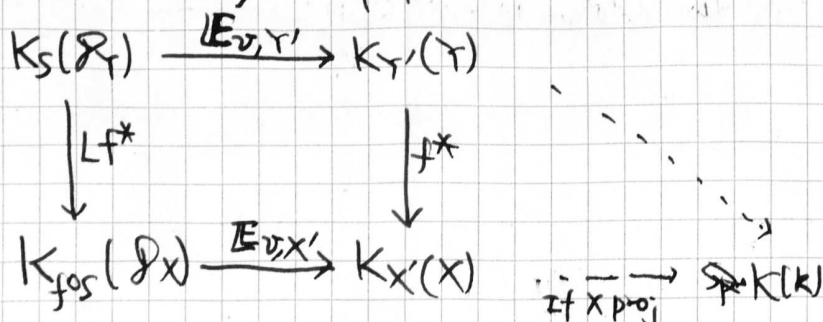


Proposition 3.1

Assume that f is étale with folloy diagram



One has a commutative diagram of spectra:



In particular if X and Y proj, and if $u \in D_{\text{perf}}^b(\mathcal{O}_X)$ then two homotopy points

proof f^* exact. The diagram in question factor as

$$\begin{array}{ccccc}
 K_S(\mathcal{O}_Y) & \longrightarrow & K_S(T^*Y) & \longrightarrow & K_{Y,1}(Y) \\
 \downarrow L^* & & \downarrow & \swarrow & \downarrow f^* \\
 K_{f_{os}}(\mathcal{O}_X) & \longrightarrow & K_{f_{os}}(T^*X) & \longrightarrow & K_{X,1}(X)
 \end{array}$$

given by $T^*Y \leftarrow T^*Y \times_T X \rightarrow X$
 $\downarrow \cong$
 T^*X

For the left square, it is enough to show the following square commutes

$$\begin{array}{ccc}
 K_S(\mathcal{O}_Y) & \longrightarrow & K_S(\text{gr } \mathcal{O}_Y) \\
 \downarrow & & \downarrow \\
 K_{f_{os}}(\mathcal{O}_X) & \longrightarrow & K_{f_{os}}(\text{gr } \mathcal{O}_X)
 \end{array}$$

defined by pullback along f and push-forward along the gr map $f^* \text{gr } \mathcal{O}_Y \cong \text{gr } \mathcal{O}_X$.

The diagram commutes since by construction of the filtered pull-back, we have $\text{gr}(f^*(u, \mathcal{F})) \cong f^*(\text{gr}(u))$ □

Now we study push-forward by finite étale maps.

Lemma For any proper morphism $f: X \rightarrow Y$ between smooth schemes, there is a commutative diagram of spectra

$$\begin{array}{ccc}
 K_F(\mathcal{O}_X) & \xrightarrow{S_f} & K_F(\mathcal{O}_Y) \\
 \downarrow \text{gr} & & \downarrow \text{gr} \\
 K(T^*X) & \xrightarrow{G} & K(T^*Y)
 \end{array}$$

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where G is induced by sending M to $R\text{pr}_{2*} \text{df}^! M[\text{df}]$, where df is the relative dimension of f .

Microlocal Riemann-Roch

Let $S \subseteq T^*X$ and $f_0 S := p_*(df^{-1}(S)) \subseteq T^*Y$, we have

$$\begin{array}{ccc} K\mathcal{F}_S(\mathcal{O}_X) & \xrightarrow{S_f} & K\mathcal{F}_{f_0 S}(\mathcal{O}_Y) \\ \downarrow \text{gr} & & \downarrow \text{gr} \\ K_S(T^*X) & \xrightarrow{G} & K_{f_0 S}(T^*Y) \end{array}$$

Corollary Let $f: X \rightarrow Y$ be a proper étale morphism with $S \subseteq T^*X$. ν : non-vanishing 1-form on $Y \setminus Y'$ with $\nu(Y \setminus Y') \cap f_0 S = \emptyset$.

Since f is étale, $(f^*\nu)(X \setminus X') \cap S = \emptyset$.

We have a commutative diagram

$$\begin{array}{ccc} K_S(\mathcal{O}_X) & \longrightarrow & K_{S'}(\mathcal{O}_Y) \\ \downarrow \mathbb{E}_{f^*\nu, X} & & \downarrow \mathbb{E}_{\nu, Y} \\ K_{X'}(X) & \longrightarrow & K_{Y'}(Y) \end{array}$$

$$\begin{array}{ccc} X' & \hookrightarrow & X \\ \downarrow p & & \downarrow f \\ Y' & \hookrightarrow & Y \end{array}$$

Enough to check $K_S(T^*X) \xrightarrow{G} K_{S'}(T^*Y)$, then we localized sequence.

$$\begin{array}{ccc} \downarrow & & \downarrow \\ K_{X'}(X) & \xrightarrow{Rf_*} & K_{Y'}(Y) \end{array}$$

Filtered \mathcal{O}_X -modules

$\mathcal{O}_0 = \mathcal{O}_X \subseteq \mathcal{O}_1 \subseteq \dots \subseteq \mathcal{O}_X$, $\mathcal{O}_i \mathcal{O}_j \subseteq \mathcal{O}_{i+j}$ and $\bigcup_i \mathcal{O}_i = \mathcal{O}_X$.

Each \mathcal{O}_i is a locally free \mathcal{O}_X -module.

A filtered \mathcal{O}_X -module consists of a pair $(\mathcal{M}, \mathcal{F}_\bullet)$, where \mathcal{M} is a \mathcal{O}_X -module and \mathcal{F}_\bullet is an increasing \mathbb{Z} -filtration of \mathcal{M} by \mathcal{O}_X -submodules such that $\mathcal{F}_i = 0$ for $i \ll 0$, $\bigcup_i \mathcal{F}_i = \mathcal{M}$ and $\mathcal{O}_i \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$.

A filtered \mathcal{O}_X -module is quasi-coherent if each \mathcal{F}_i is a quasi-coherent \mathcal{O}_X -module.

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§4 ϵ -factor supported on points

Idea Patel's epsilon factor $k_S(\mathcal{O}_X) \xrightarrow{E_{\nu, \gamma}} k_{\gamma}(X)$ is expected to be supported on a closed subset Y of X of codimension 1.

Using several 1-forms at once, one can replace Y by a smaller closed subset Z , even of codimension $d = \dim X$.

This observation is due to Michael Graecheing

Notation I finite non-empty ordered set.

$$\Delta^I := \frac{\mathbb{A}^I}{\langle \sum \lambda_i - 1 \rangle} \text{ affine space of } \dim |I| - 1, \text{ defined by } \sum \lambda_i = 1.$$

Let X be a smooth d -dimensional scheme over a field k of characteristic 0.

$$\underbrace{Z \subseteq X}_{\text{closed}} \longleftarrow U = X \setminus Z. \quad S \subseteq T^*X \text{ conical closed.}$$

Consider an open covering $U = \bigcup_{i=1}^m U_i$ and for each $i=1, \dots, m$, a nowhere vanishing 1-form $\nu_i \in \Omega_X^1(U_i)$ such that for each ordered subset $I = \{i_1 < \dots < i_\ell\} \subseteq \{1, \dots, m\}$, we have for $U_{i_1, \dots, i_\ell} = \bigcap_{j=1}^{\ell} U_{i_j}$ that the image of the morphism

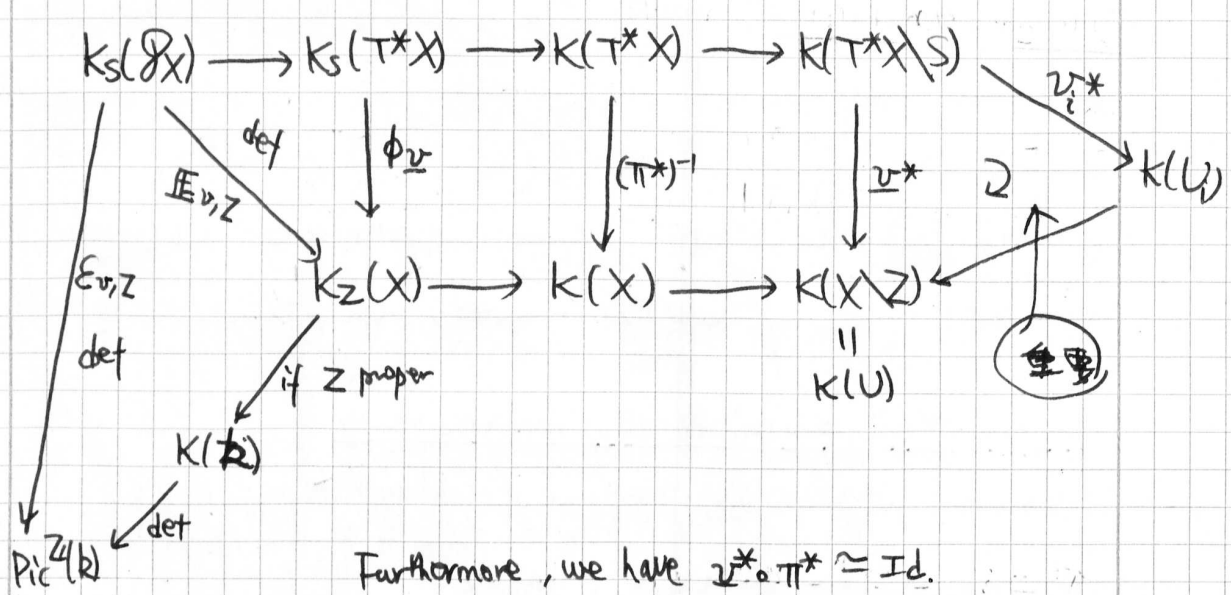
$$\begin{aligned} \nu_\Delta : U_{i_1, \dots, i_\ell} \times \Delta^I &\longrightarrow T^*X \\ \nu_\Delta(\lambda_1, \dots, \lambda_\ell) &= \sum_{j=1}^{\ell} \lambda_j \nu_{i_j}^{\leftarrow u} \end{aligned}$$

such that $\text{Im}(\nu_\Delta) \cap S = \emptyset$.

Lemma + Definition There exists a morphism

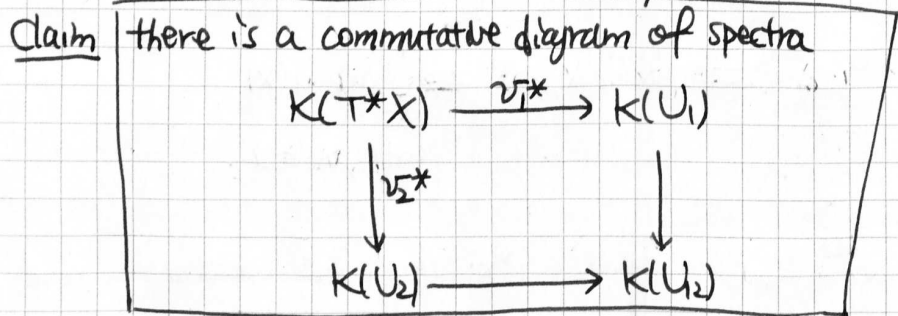
$$(\mathcal{Z})^* : k(T^*X \setminus S) \longrightarrow k(X \setminus Z)$$

such that for all $i=1, \dots, m$, we have a commutative diagram



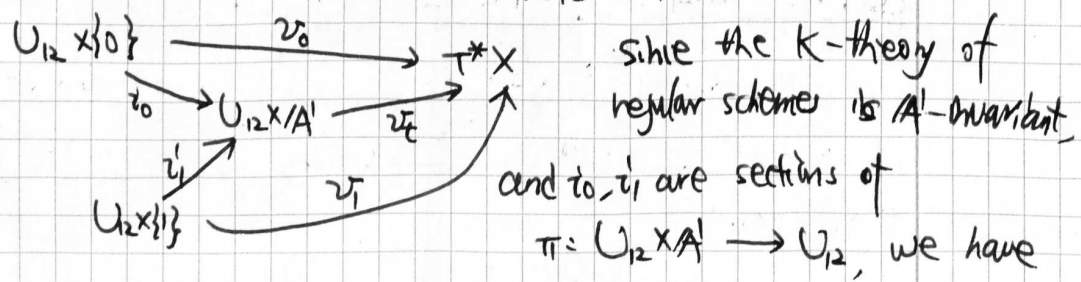
Sketch of the proof for $m=2$

The idea is to show that on $U_2 = U_1 \cap U_2$, one can construct a linear homotopy $\nu_t : U_2 \times \mathbb{A}^1 \rightarrow T^*U_2$ between the sections $\nu_1|_{U_2}$ and $\nu_2|_{U_2}$ by $\nu_t = (1-t)\nu_1 + t\nu_2$ for $t \in \mathbb{A}^1$. In this step, we use that ν_1 and ν_2 are linearly independent on U_2 .



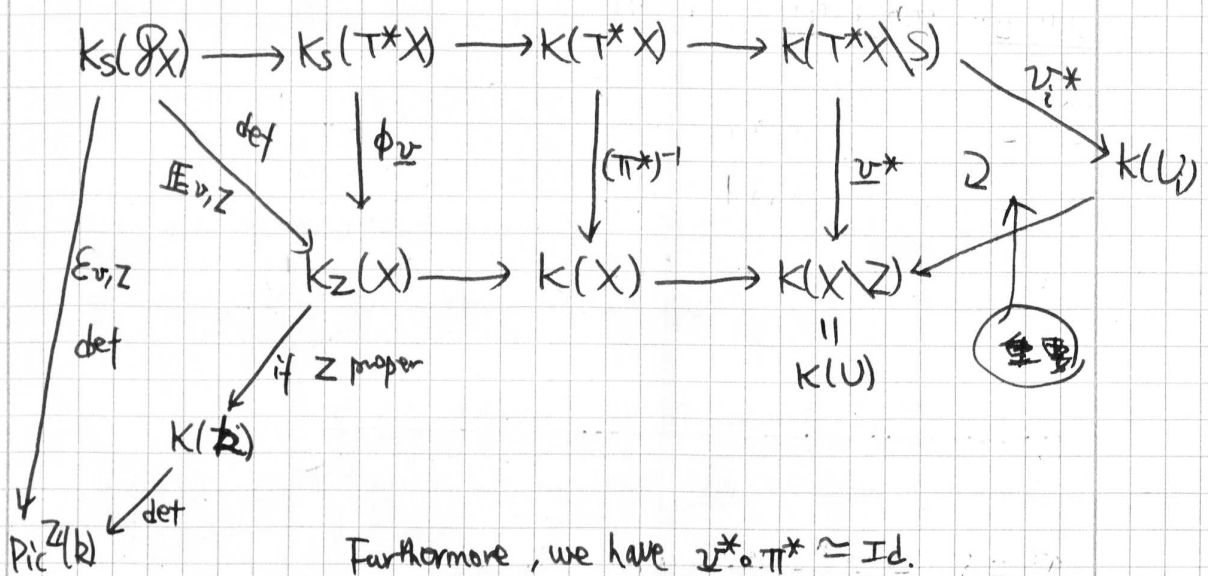
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proof of claim we show $(\nu_1|_{U_2})^* = (\nu_2|_{U_2})^*$.



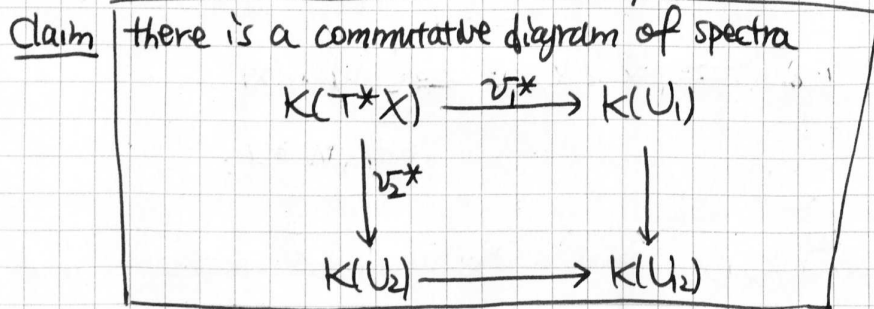
that the pullback along i_0 and i_1 are inverse to π^* ,
 therefore $\nu_0^* \simeq i_0^* \circ \nu_t^* \simeq (\pi^*)^{-1} \circ \nu_t^* \simeq i_1^* \circ \nu_t^* \simeq \nu_1^*$

Remark If $\dim Z = 0$ then $E_{\nu, Z}(U) = \bigotimes_{i=1}^m E_{\nu, Z}(U_i)$



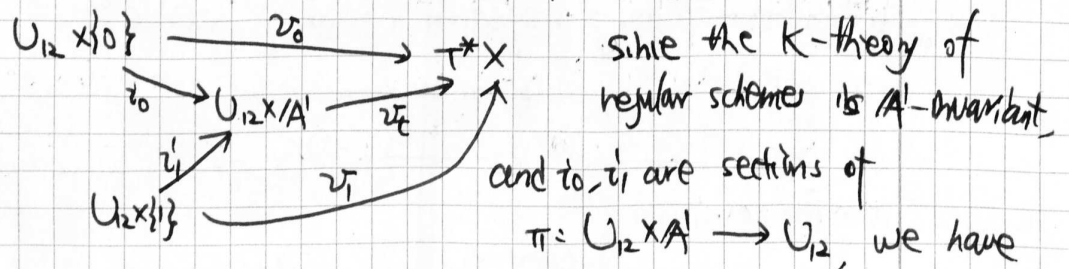
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Remark If $\dim Z = 0$, then $E_{v,Z}(U) = \bigotimes_{z \in \pi_0(Z)} E_{v,z}(U)$.

12 § 5 Twist formula

X projective smooth.

$F \in \text{D}_{\text{hol}}(\mathcal{O}_X)$.

$S \in \text{D}_{\text{hol}}(\mathcal{O}_X)$ with $\text{SS}(S) \subseteq T^*X \Rightarrow S$ vector bundle with connection.

We show $E_{\text{dr}}(X, S \otimes F) \cong \langle \det S, \text{CCF} \rangle$.

We first recall Levine's homotopy coniveau tower

X/k smooth. $K(X) = k$ -theory spectrum of coherent sheaves on X .

$\Delta^n = \text{Spec} \frac{k[t_0, \dots, t_n]}{(\sum t_i - 1)}$ usual n -simplex.

A face of Δ^n is a closed subscheme defined by equations of the form

$$t_{i_1} = \dots = t_{i_s} = 0.$$

Then one defines $K^{(q)}(X, p) := \varinjlim_W K_W(X \times \Delta^p)$ where the homotopy limit is taken over closed subschemes $W \subseteq X \times \Delta^p$ such that

$\text{codim}_{X \times F}(W \cap X \times F) \geq q$ for all faces $F \subseteq \Delta^p$.

We put $K^{(q)}(X) := K^{(q)}(X, 0) = \varinjlim_W K_W(X)$
 $\text{codim}_X W \geq q$.

The spectra $K^{(q)}(X, p)$ form a \mathbb{S} -simplicial spectrum, and we let $K^{(q)}(X, -)$ denote the corresponding total spectrum.

Moreover, one has a tower of spectra (homotopy coniveau tower)

$$\dots \rightarrow K^{(2)}(X, -) \rightarrow K^{(1)}(X, -) \rightarrow \dots \rightarrow K^{(0)}(X, -)$$

with augmentation maps $\eta_q: K^{(q)}(X) \rightarrow K^{(q)}(X, -)$.

with the following properties proved by Levine:

(1) pull-back by smooth map

(2) $K(X) \rightarrow K^{(0)}(X) \rightarrow K^{(0)}(X, -)$ is a weak equivalence.
 the composition

(3) The cofibers $K^{(p+1)}(X, -)$ of the homotopy coniveau tower are naturally weak equivalent to Bloch's higher Chow group cycle complex. In particular, there is a

$$CH^d(X) \longrightarrow \pi_0(K^{(d)}(X, -)) \text{ if } d = \dim X.$$

(4) tensor product induces natural pairings

$$K^{(d)}(X, -) \wedge K^{(d')}(X, -) \longrightarrow K^{(d+d')}(X, -).$$

In particular $K(X) \wedge K^{(d)}(X, -) \longrightarrow K^{(d)}(X, -).$

Now we always assume that X is smooth projective over k

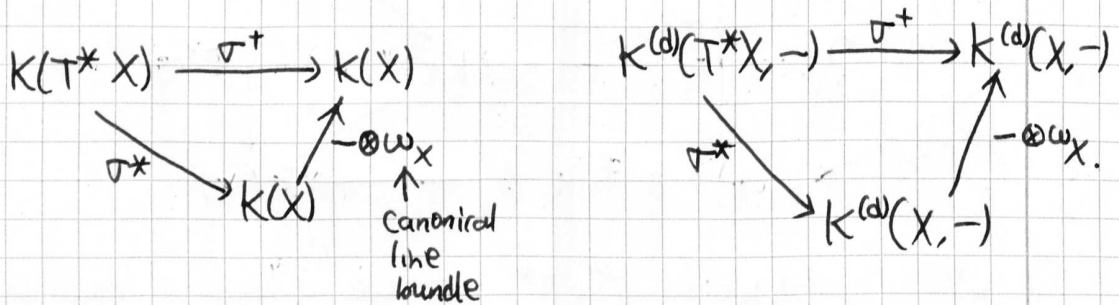
Patal's result $g_{\mathbb{A}^1}: K_S(\mathcal{P}_X) \longrightarrow K_S(T^*X) \quad S \subseteq T^*X$

take limit, we get $K_{hol}(\mathcal{P}_X) \longrightarrow K^{(d)}(T^*X).$

For a holonomic \mathcal{P}_X -module \mathcal{F} , we set $E_{DR}(X, \mathcal{F}) = \det R\Gamma_{DR}(X, \mathcal{F}) \in \text{Pic}^{\mathbb{Z}}(k)$

Consider $CC^k: K_{hol}(\mathcal{P}_X) \xrightarrow{\epsilon} K^{(d)}(T^*X) \longrightarrow K^{(d)}(T^*X, -)$

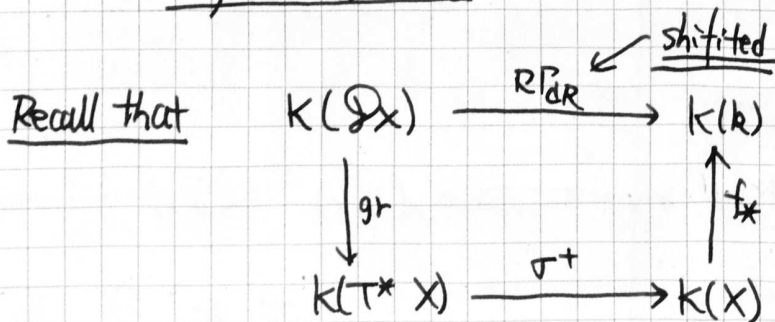
Twisted pullback by σ^* : $\mathcal{F}: X \rightarrow T^*X$ zero section



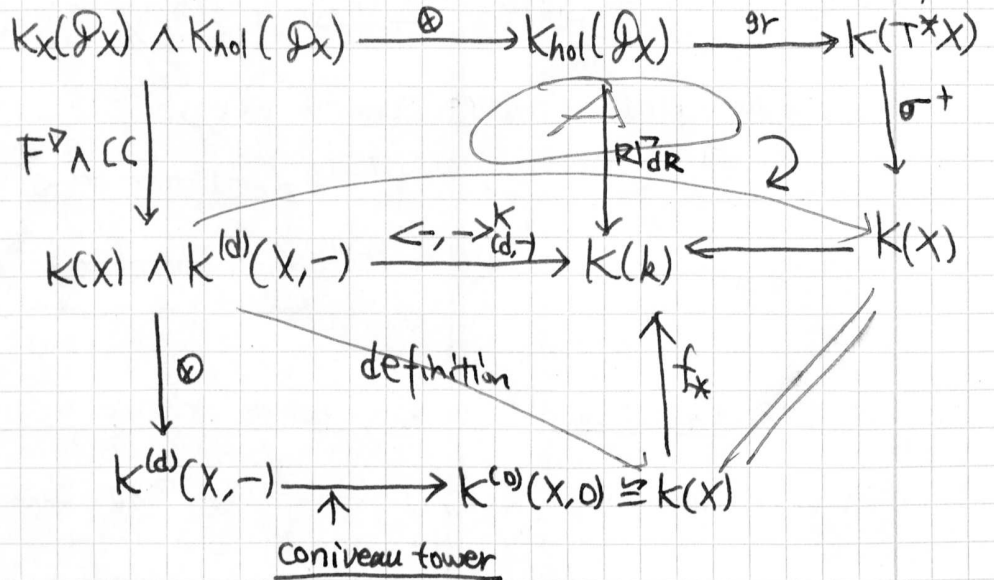
We set $CC := \sigma^+ \circ CC^k$

let $F^D: K_X(\mathcal{P}_X) \longrightarrow K(X)$ forgetting the \mathcal{P}_X -module structure

\uparrow
 $S \subseteq T^*X$
regards \mathcal{O}_X -module

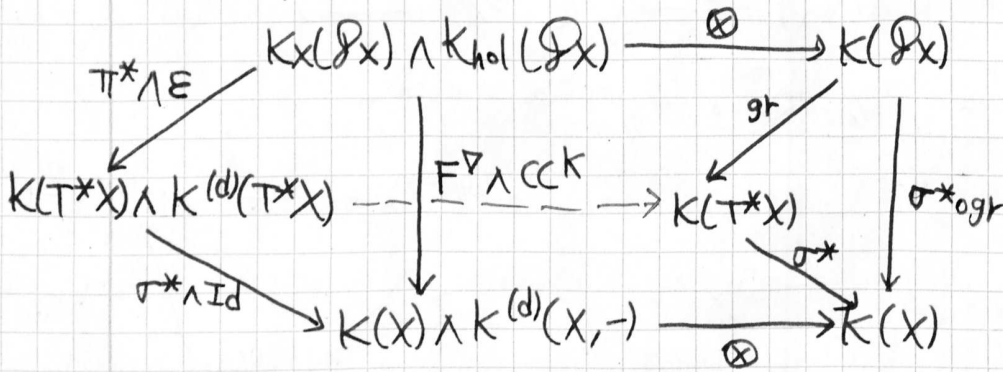


Theorem (Abe-Patel) The following diagram commutes up to homotopy equivalence

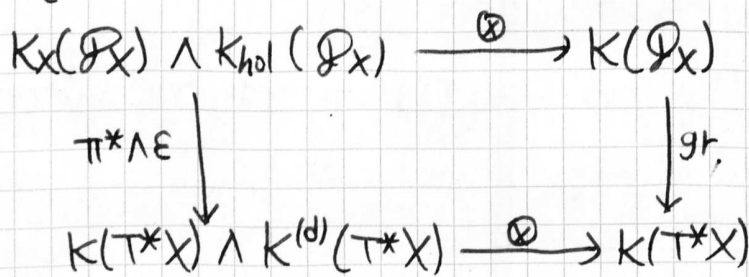


Proof We only need to show (A) is commutative.

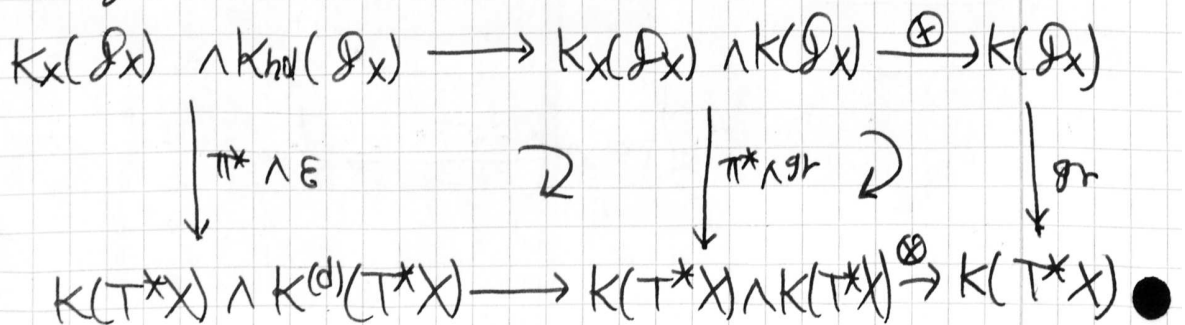
Since σ^+ commutes with \otimes and argumentation, we reduce to



As F^∇ is homotopic to $\sigma^* \circ \pi^*$, we are reduced to show that the following diagram commutes



By def, this diagram factors as



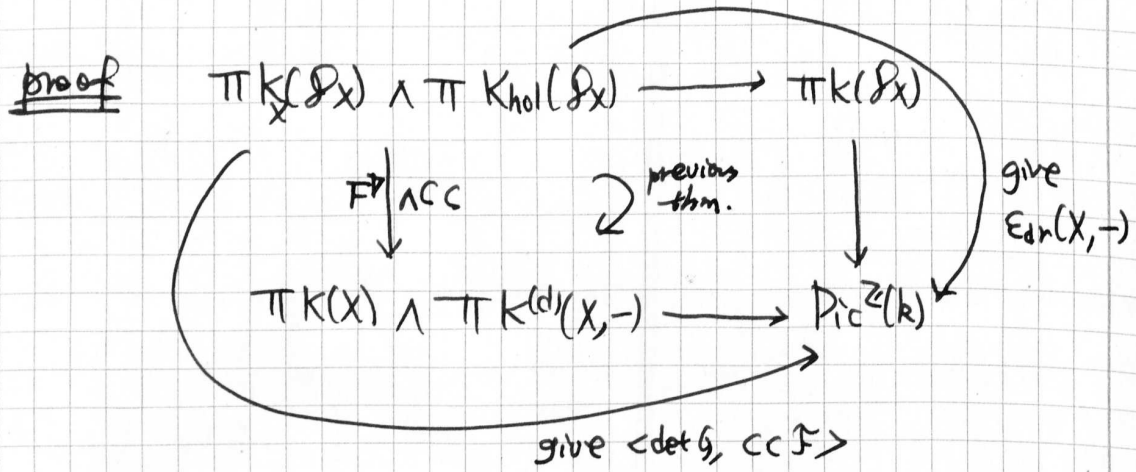
Corollary (twist formula)

- \mathcal{F} holonomic \mathcal{O}_X -module on projective smooth scheme X
- \mathcal{G} vector bundle with connection. then forgetting the connection, \mathcal{G} give rises to a natural object $\det \mathcal{G} \in \pi K(X)$.
- \mathcal{F} gives rise to an object of the Picard groupoid associated to $\text{Khol}(\mathcal{O}_X)$, hence an object of $\pi K^{(d)}(X, -)$ via the morphism $CC: \text{Khol}(\mathcal{O}_X) \rightarrow K^{(d)}(X, -)$.
- We denote the corresponding object by $CC \mathcal{F} \in \pi K^{(d)}(X, -)$.

— Applying the pairing $\langle -, - \rangle_{cc}: \pi K(X) \wedge \pi K^{(d)}(X, -) \rightarrow \text{Pic}^{\mathbb{Z}}(k)$
 we get $\langle \det \mathcal{G}, CC \mathcal{F} \rangle \in \text{Pic}^{\mathbb{Z}}(k)$.

Corollary One has a natural isomorphism $\text{zh } \text{Pic}^{\mathbb{Z}}(k)$

$$\text{Etr}(X, \mathcal{G} \otimes \mathcal{F}) \cong \langle \det \mathcal{G}, CC \mathcal{F} \rangle$$



Theorem $f \in \text{End}(\mathcal{F}), g \in \text{End}(\mathcal{G}). f \otimes g \in \text{End}(\text{RP}_{\text{tr}}(X, \mathcal{F} \otimes \mathcal{G}))$

put $\text{Etr}(X, \mathcal{F} \otimes \mathcal{G}; f \otimes g) = \text{Tr}(f \otimes g | \text{RP}_{\text{tr}}(X, \mathcal{F} \otimes \mathcal{G})) \in k^X$.

$r_{\mathcal{G}}$ = generic rank of \mathcal{G} , i.e., image of \mathcal{G} by the map $\text{Tr}(\text{Pic}^{\mathbb{Z}}(X)) \rightarrow \mathbb{Z}$.

Then
 $\text{Etr}(X, \mathcal{F} \otimes \mathcal{G}; f \otimes g) = \text{Etr}(X, \mathcal{F}; f)^{r_{\mathcal{G}}} \times \langle \det \mathcal{G}, CC \mathcal{F} \rangle(g, \text{Id})$

Trace of $g \otimes \text{Id} \in \text{End}(\langle \det \mathcal{G}, CC \mathcal{F} \rangle)$

